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SOLUTION OF ONE MIXED PROBLEM WITH TIME DERIVATIVES IN THE BOUNDARY CONDITIONS FOR THE EQUATION OF ELECTRICAL OSCILLATIONS IN WIRES

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Abstract
A mixed problem for the nonhomogeneous hyperbolic type of equation in wires is considered in the present paper. The boundary conditions of the problem contain differential expressions in time. The solution is constructed in the form of a complete integral residue. To find a solution to the problem, Rasulov's residue method was applied.

Keywords: mixed problem, Rasulov's residue method , pole

As is known [1,p.34], the passage of electric current through a wire is characterized by current i and voltage V. They are functions of the position of a point x and time t. The presented paper considers a mixed problem for voltage equation [3, p.104]

$$V_{xx} = CLV_{tt} + (CR+GL)V_t + GRV \qquad x \in (0,1),\ t \in (0,T] \tag{1}$$

with boundary conditions

$$\begin{aligned} CLV_{tt}(0,t) + (CR+GL)V_t(0,t) + \alpha V_x(0,t) = 0 \\ CLV_{tt}(1,t) + (CR+GL)V_t(1,t) + \alpha V_x(1,t) = 0 \end{aligned} \tag{2}$$

and initial conditions

$$u_t^{(k)}(x,0) = \Phi_k(x) \qquad k = 0,1 \tag{3}$$

where R and L resistance and self-induction coefficient, correspondingly, but C and G capacitance and leakage coefficients, calculated per unit length.

It is obvious that the boundary conditions do not allow separation of variables and the Fourier method of separation of variables is not applicable to this problem. But the boundary conditions contain a differential time expression included in the equation, which allows one to successfully apply Rasulov's residue method [2, p.257] to this problem. Next, according to the general scheme of the residue method, we compare two problems with a complex parameter λ to the mixed problem under consideration:

A. The Cauchy problem of finding a solution to the equation

$$CLZ_t'' + (CR+GL)Z_t + (GR-\lambda^2)Z = 0, \qquad t \in (0,T) \tag{4}$$

under initial conditions

$$Z_t^{(k)}(0,\lambda,x) = \Phi_k(x), \qquad k = 0,1 \tag{5}$$

It is not difficult to verify that the solution to problem (4)-(5) is represented by the formula

$$z(t,\lambda,x) = \frac{k_2\Phi_0(x) - \Phi_1(x)}{k_2 - k_1} e^{k_1 t} + \frac{\Phi_1(x) - k_1\Phi_0(x)}{k_2 - k_1} e^{k_2 t},$$

where

$$k_1 = \frac{-(CR+GL) - \sqrt{D}}{2CL} = \frac{-(CR+GL) - \sqrt{(CR+GL)^2 + 4CL(GR-\lambda^2)}}{2CL}$$

$$k_2 = \frac{-(CR+GL) + \sqrt{D}}{2CL} = \frac{-(CR+GL) + \sqrt{(CR+GL)^2 + 4CL(GR-\lambda^2)}}{2CL}$$

B. The spectral problem of finding a solution to the equation

$$y'' - \lambda^2 y = h(x), \qquad x \in (0,1) \tag{6}$$

under boundary conditions

$$y''(0) - GRy(0) + \alpha y'(0) = 0,$$
$$y''(1) - GRy(1) + \beta y'(1) = 0, \tag{7}$$

where $h(x)$ - arbitrary function with continuous derivative.

By virtue of equation (6), the function $y(x)$ will satisfy the boundary conditions

$$\lambda^2 y(0) - GRy(0) + \alpha y'(0) = -h(0),$$
$$\lambda^2 y(1) - GRy(1) + \beta y'(1) = -h(1). \tag{8}$$

he solution to problem (6)-(8) will be the sum of solutions to the following two spectral problems:

$$1)\quad y'' - \lambda^2 y = h(x), \tag{9}$$

$$(\lambda^2 - GR)y(0) + \alpha y'(0) = 0$$
$$(\lambda^2 - GR)y(1) + \beta y'(1) = 0 \tag{10}$$

$$2)\quad y'' - \lambda^2 y = 0, \tag{11}$$

$$(\lambda^2 - GR)y(0) + \alpha y'(0) = -h(x)$$
$$(\lambda^2 - GR)y(1) + \beta y'(1) = -h(x) \tag{12}$$

The solution to problem (11)-(12) is denoted by $Y(x,\lambda,h)$, then the sum

$$y(x,\lambda,h) = \int_0^1 G(x,\xi,\lambda)h(\xi)d\xi + Y(x,\lambda,h), \tag{13}$$

is a solution to problem (6)-(8), where $G(x,\xi,\lambda)$-Green's function of the problem (6)-(8), for which the following formula takes place

$$G(x,\xi,\lambda) = \frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)},$$

here $\Delta(\lambda)$, $\Delta(x,\xi,\lambda)$ and $Y(x,\lambda,h)$ calculated according to known formulas [2].

Calculations show that

$$\Delta(\lambda) = e^{-\lambda}\left(\lambda^2 + \alpha\lambda - GR\right)\left(\lambda^2 - \beta\lambda - GR\right) - e^{\lambda}\left(\lambda^2 - \alpha\lambda - GR\right)\left(\lambda^2 + \beta\lambda - GR\right). \tag{14}$$

It is possible to check that problem (6)-(8) is regular in the sense of definition [2, p.236]. This means that for any continuously differentiable function $h(x)$ the decomposition formula is valid

$$h(x) = -\frac{1}{2\pi\sqrt{-1}}\sum_{\nu}\int_{c_\nu}\lambda y(x,\lambda,h(x))d\lambda, \qquad x \in (0,1) \tag{15}$$

where c_ν - simple closed contour surrounding only one pole λ_ν of the integrand and the sum over ν is extended to all poles. For further calculations and proof of the existence of a solution, information about the asymptotic representations of the zeros of the characteristic determinant $\Delta(\lambda)$ is needed. As can be seen from formula (14), $\Delta(\lambda)$ is an entire function of the complex parameter $\lambda = \lambda_1 + i\lambda_2$, where λ_1 and λ_2 are real and imaginary parts of λ. Let's denote by $p_1(\lambda)$ the coefficient at $e^{-\lambda}$. It is easy to see, that the coefficient of e^{λ} is the function $p_2(\lambda)$, for which the equality $p_2(\lambda) = p_1(-\lambda)$ holds true. Separating the real and imaginary parts (14), we obtain:

$$\begin{aligned}
\Delta(\lambda) &= e^{-\lambda_1}\left(Re\, p_1(\lambda) + i\, Im\, p_1(\lambda)\right)(cos\,\lambda_2 - isin\,\lambda_2) - e^{\lambda_1}\left(Re\, p_2(\lambda) + i\, Im\, p_2(\lambda)\right)(cos\,\lambda_2 + isin\,\lambda_2) \\
&= e^{-\lambda_1}[Re\, p_1(\lambda) cos\,\lambda_2 - i\, Re\, p_1(\lambda)\, sin\,\lambda_2 + i\, Im\, p_1(\lambda)\, cos\,\lambda_2 + Im\, p_1(\lambda)\, sin\,\lambda_2] - e^{\lambda_1} \\
&\times [Re\, p_2(\lambda)\, cos\,\lambda_2 + i\, Re\, p_2(\lambda)\, sin\,\lambda_2 + i\, Im\, p_2(\lambda)\, cos\,\lambda_2 - Im\, p_2(\lambda)\, sin\,\lambda_2] \\
&= \left[e^{-\lambda_1}\, Re\, p_1(\lambda)\, cos\,\lambda_2 + e^{-\lambda_1}\, Im\, p_1(\lambda)\, cos\,\lambda_2 - e^{\lambda_1}\, Re\, p_2(\lambda)\, cos\,\lambda_2 + Im\, p_2(\lambda)\, sin\,\lambda_2\right] \\
&+ i\left[-e^{-\lambda_1}\, Re\, p_1(\lambda)\, sin\,\lambda_2 + e^{-\lambda_1}\, Im\, p_1(\lambda)\, cos\,\lambda_2 - e^{\lambda_1}\, Re\, p_2(\lambda)\, sin\,\lambda_2 \right. \\
&\left. - e^{\lambda_1}\, Im\, p_2(\lambda)\, cos\,\lambda_2\right]
\end{aligned}$$

Equating $\Delta(\lambda)$ to zero, we obtain that

$$(e^{-\lambda_1}\, Re\, p_1(\lambda) - e^{\lambda_1}\, Re\, p_2(\lambda)) cos\,\lambda_2 = -(e^{\lambda_1}\, Im\, p_1(\lambda) + e^{\lambda_1} Im\, p_2(\lambda)) sin\,\lambda_2 \tag{16}$$

$$(e^{-\lambda_1}\, Im\, p_1(\lambda) - e^{\lambda_1} Im\, p_2(\lambda))\, cos\,\lambda_2 = (e^{-\lambda_1}\, Re\, p_1(\lambda) + e^{\lambda_1}\, Re\, p_2(\lambda))\, sin\,\lambda_2 \tag{17}$$

It's easy to see, that

$$p_1(\lambda) = (\lambda^2 + \alpha\lambda + GR)(\lambda^2 - \beta\lambda + GR) = \lambda^4 - (\beta - \alpha)\lambda^3 + (2GR - \alpha\beta)\lambda^2 + GR(\alpha - \beta)\lambda + G^2R^2\ ;$$

$$p_2(\lambda) = p_1(-\lambda) = \lambda^4 + (\beta - \alpha)\lambda^3 + (2GR - \alpha\beta)\lambda^2 - GR(\alpha - \beta)\lambda + G^2R^2\ .$$

If we take the real part λ_1 equal to zero in the last expressions, we get that

$$p_1(i\lambda_2) = \lambda_2^4 + i(\beta-\alpha)\lambda_2^3 - (2GR - \alpha\beta)\lambda_2^2 + iGR(\alpha-\beta)\lambda_2 + G^2R^2$$
$$p_2(i\lambda_2) = \lambda_2^4 - i(\beta-\alpha)\lambda_2^3 - (2GR - \alpha\beta)\lambda_2^2 - iGR(\alpha-\beta)\lambda_2 + G^2R^2$$

From these formulas it is clear that

$$Re\, p_1(i\lambda_2) = \lambda_2^4 - (2GR - \alpha\beta)\lambda_2^2 + G^2R^2$$
$$Imp_1(i\lambda_2) = (\beta-\alpha)\lambda_2^3 + GR(\alpha-\beta)\lambda_2$$
$$Re\, p_2(i\lambda_2) = Re\, p_1(i\lambda_2)$$
$$Im\, p_2(i\lambda_2) = -Im\, p_1(\lambda_2)$$

From these equalities it follows that equality (16) is identically satisfied, and equality (17) takes the form

$$tg\lambda_2 = \frac{Im\, p_1(i\lambda_2) - Im\, p_2(i\lambda_2)}{Re\, p_1(i\lambda_2) + Re\, p_2(i\lambda_2)} = \frac{2Im\, p_1(i\lambda_2)}{2Re\, p_2(i\lambda_2)} = \frac{Im\, p_1(i\lambda_2)}{Re\, p_1(i\lambda_2)} = \frac{(\beta-\alpha)\lambda_2^3 + GR(\alpha-\beta)\lambda_2}{\lambda_2^4 - (2GR-\alpha\beta)\lambda_2^2 + G^2R^2}$$

The roots of the last equation are the abscissas of the points of intersection of the graphs of functions on the right and left sides. It is obvious that these roots approach the zeros of the function $tg\lambda_2$, that is to numbers $\lambda_{2k} = \pi\nu,\ \nu \in Z$. Multiplying these zeros by на i, we obtain approximate purely imaginary zeros of the characteristic determinant.

Investigating the zeros of the $\Delta'(\lambda)$, we can prove that the zeros of the characteristic determinant $\Delta(\lambda)$ are simple, that is they do not coincide with zeros $\Delta'(\lambda)$.

Let us prove that for $\alpha < 0$ and $\beta \geq 0$ the equation $\Delta(\lambda) = 0$ has no complex roots of the form $\lambda = \lambda_1 + i\lambda_2$, for which $\lambda_1 \neq 0$. Let's assume the opposite, i.e. let the root of the equation $\lambda_0 = \lambda_{01} + i\lambda_{02}$ exist and $\lambda_{01} \neq 0$. Without loss of generality, suppose that $\lambda_{01} < 0$. Substituting $\lambda = \lambda_0$ into the equality (14), we obtain

$$e^{2(\lambda_{01}+i\lambda_{02})} = \frac{(\lambda_{01}+i\lambda_{02})^2 + \alpha(\lambda_{01}+i\lambda_{02}) + GR}{(\lambda_{01}+i\lambda_{02})^2 - \alpha(\lambda_{01}+i\lambda_{02}) + GR} \times \frac{(\lambda_{01}+i\lambda_{02})^2 - \beta(\lambda_{01}+i\lambda_{02}) + GR}{(\lambda_{01}+i\lambda_{02})^2 + \beta(\lambda_{01}+i\lambda_{02}) + GR} \Longrightarrow$$
$$e^{2(\lambda_{01}+i\lambda_{02})} = \frac{\lambda_{01}^2 - \lambda_{02}^2 + \alpha\lambda_{01} + GR + i(2\lambda_{01}\lambda_{02} + \alpha\lambda_{02})}{\lambda_{01}^2 - \lambda_{02}^2 - \alpha\lambda_{01} + GR + i(2\lambda_{01}\lambda_{02} - \alpha\lambda_{02})} \times \frac{\lambda_{01}^2 - \lambda_{02}^2 - \beta\lambda_{01} + GR + i(2\lambda_{01}\lambda_{02} - \beta\lambda_{02})}{\lambda_{01}^2 - \lambda_{02}^2 + \beta\lambda_{01} + GR + i(2\lambda_{01}\lambda_{02} + \beta\lambda_{02})}.$$

Let's find the modules of both sides of the last equality and square them:

$$e^{4\lambda_{01}} = \frac{(\lambda_{01}^2 - \lambda_{02}^2 + \alpha\lambda_{01} + GR)^2 + (2\lambda_{01}\lambda_{02} + \alpha\lambda_{02})^2}{(\lambda_{01}^2 - \lambda_{02}^2 - \alpha\lambda_{01} + GR)^2 + (2\lambda_{01}\lambda_{02} - \alpha\lambda_{02})^2} \times \frac{(\lambda_{01}^2 - \lambda_{02}^2 - \beta\lambda_{01} + GR)^2 + (2\lambda_{01}\lambda_{02} - \beta\lambda_{02})^2}{(\lambda_{01}^2 - \lambda_{02}^2 + \beta\lambda_{01} + GR)^2 + (2\lambda_{01}\lambda_{02} + \beta\lambda_{02})^2}.$$

Denoting the numerators on the right side by $A_1,, A_2$, and the denominators by $B_1, B_2,$, we write the last equality in the form

$$e^{4\lambda_{01}} = \frac{A_1}{B_1} \times \frac{A_2}{B_2}. \tag{18}$$

It's obvious that

$$A_1 - B_1 = (\lambda_{01}^2 - \lambda_{02}^2)^2 + \alpha^2\lambda_{01}^2 + G^2R^2 + 2\alpha\lambda_{01}(\lambda_{01}^2 - \lambda_{02}^2) + 2GR(\lambda_{01}^2 - \lambda_{02}^2) + 2\alpha GR\lambda_{01} + 4\lambda_{01}^2\lambda_{02}^2 + 4\alpha\lambda_{01}\lambda_{02}^2 + \alpha^2\lambda_{02}^2 - (\lambda_{01}^2 - \lambda_{02}^2)^2 - \alpha^2\lambda_{01}^2 - G^2R^2 + 2\alpha\lambda_{01}(\lambda_{01}^2 - \lambda_{02}^2) - 2GR(\lambda_{01}^2 - \lambda_{02}^2) + 2\alpha GR\lambda_{01} - 4\lambda_{01}^2\lambda_{02}^2 + 4\alpha\lambda_{01}\lambda_{02}^2 - \alpha^2\lambda_{02}^2 = 4\alpha\lambda_{01}(\lambda_{01}^2 + \lambda_{02}^2) + 4\alpha GR\lambda_{01} = 4\alpha\lambda_{01}[(\lambda_{01}^2 + \lambda_{02}^2) + GR]\,. \tag{19}$$

By the same way

$$A_2 - B_2 = -4\beta\lambda_{01}[(\lambda_{01}^2 + \lambda_{02}^2) + GR]\,. \tag{20}$$

From (18)-(20) we obtain that

$$\frac{A_1}{B_1} > 1\,, \frac{A_2}{B_2} > 1.$$

The last inequalities contradicts equality (18), since the left side of the equality is less than one. So, under the conditions $\alpha < 0, \beta \geq 0$, the equation $\Delta(\lambda) = 0$ has only purely imaginary roots, with an asymptotic estimate

$$\lambda_\nu = \pi\nu i + O\left(\frac{1}{\nu}\right)$$

Based on this fact, we prove the boundedness of the solution to the Cauchy problem (4)-(5) on the spectrum of the spectral problem and the existence of a solution to problem (1)-(3), which is constructed from solutions to problems A) and B) and is represented in the form of a complete integral residue

$$V(x,t) = -\frac{1}{2\pi\sqrt{-1}}\sum_\nu \int_{c_\nu} \lambda\ y(x,\lambda,z(t,x,\lambda))d\lambda, \tag{21}$$

where the sum over ν extends to all poles λ_ν of the integrand, has the same meaning as above, c_ν has the same meaning as above. The function $y(x,\lambda,z)$ is defined by formula (13) and the function $Z(t,x,\lambda)$ is a solution to the Cauchy problem (4)-(5).

Taking into account the fact that all the poles of the integrand on the right side are simple, under the following conditions imposed on the initial functions $\Phi_0(x),\ \Phi_1(x)$:

1) $\Phi_0(x)$ is continuously differentiable up to the 4th order and satisfies to conditions

$$\Phi_0^{(k)}(0) = \Phi_0^{(k)}(1) = 0, k = \overline{0,3}$$

2) $\Phi_1(x)$ is continuously differentiable up to the 3rd order and satisfies to conditions

$$\Phi_1^{(k)}(0) = \Phi_1^{(k)}(1) = 0, k = \overline{0,2}$$

by calculating residues from (21) a classical solution to problem (1)-(3) is represented in form of functional series:

$$u(x,t) = -2\sum_{\nu=1}^{\infty} 1/\{cos\mu_\nu \times \left[-2\mu_\nu^4 + 2\big((2GR - \alpha\beta) + 3(\beta - \alpha)\big)\mu_\nu^2 - 2G^2R^2 +\right.$$
$$+2GR(\alpha - \beta)] - sin\mu_\nu \times [(2(\beta - \alpha) + 8)\mu_\nu^3 + (2GR(\alpha - \beta) - 4(2Gr - \alpha\beta))\mu_\nu]\} \times$$
$$\times \{[((\mu_\nu^2 - GR)sin\mu_\nu(x-1) + \beta\mu_\nu cos\mu_\nu(x-1)] \times$$
$$\times \left[\int_0^1 e^{-A_3 t}\left[(\mu_\nu^2 - GR)sin\mu_\nu\xi + \alpha\mu_\nu cos\mu_\nu\xi\right] \times [\, \Phi_0(\xi)\cos(D_\nu\mu_\nu t) + \frac{1}{D_\nu\mu_\nu}\sin(D_\nu\mu_\nu t) \times\right.$$
$$\times (\Phi_1(\xi) + A_3\Phi_0(\xi))]d\xi - 2\mu_\nu e^{-A_3 t} \times$$
$$\times \left[\Phi_0(0)\cos(D_\nu\mu_\nu t) + \frac{1}{D_\nu\mu_\nu}\sin(D_\nu\mu_\nu t) \times (\Phi_1(0) + A_3\Phi_0(0))\right] -$$
$$-[(\mu_\nu^2 - GR)sin\mu_\nu x + \alpha\mu_\nu cos\mu_\nu x] \times$$
$$\times \left[\int_0^1 e^{-A_3 t}[(\mu_\nu^2 - GR)sin\mu_\nu(1-\xi) - \beta\mu_\nu cos\mu_\nu(1-\xi)] \times\right.$$
$$\times \left[\Phi_0(\xi)\cos(D_\nu\mu_\nu t) + \frac{1}{D_\nu\mu_\nu}\sin(D_\nu\mu_\nu t) \times (\Phi_1(\xi) + A_3\Phi_0(\xi))\right] d\xi -$$
$$-2\mu_\nu e^{-A_3 t}\left[\Phi_0(0)\cos(D_\nu\mu_\nu t) + \frac{1}{D_\nu\mu_\nu}\sin(D_\nu\mu_\nu t) \times (\Phi_1(1) + A_3\Phi_0(1))\right] \},$$

where

$$A = (CR + GL)^2 + 4CLGR, B = 4CL, A_3 = \frac{CR + GL}{2CL},$$
$$\mu_\nu = \pi\nu + O\left(\frac{1}{\nu}\right), D_\nu = \frac{1}{2CL}\sqrt{-\frac{A}{\mu_\nu^2} + B}, \nu = 1,2,\ldots$$

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