MATHEMATICAL SCIENCES

FILTRATION LOSSES FROM CHANNELS AND STREAM BEDS OF THE IRRIGATION SYSTEM

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Abstract

The solution of problems of filtration from channels and stream beds is considered. Depending on the geological structure of the soils in which filtration occurs, and other natural conditions (precipitation, evaporation, etc.), the problems of filtration without backwater (in homogeneous and inhomogeneous soil) from non-colmated channels and stream beds of arbitrary cross section are considered. It is assumed that the fluid is incompressible, its motion in the ground is steady and obeys Darcy's law (linear filtration). Capillary uplift and inertial forces are not taken into account. It is also assumed that infiltration and evaporation are absent (evaporation can be considered as negative infiltration).

Keywords: filtration from channels, heterogeneous soil, arbitrary cross section

Introduction

At present, the problem of efficient use of natural resources and nature protection is of particular importance. One of the issues of this problem is the rational use of water resources, the protection of water basins from pollution, as well as land from salinization and oxidation during irrigation. In this regard, the importance of solving problems of filtration from channels and stream beds has increased.

Problem Statement

The problem of filtering from a channel or a stream bed of an arbitrary cross section, using the method of successive conformal mappings (proposed by M.A.Lavrentiev and developed in detail by P.F.Filchakov), can be reduced to the corresponding problem of filtering from a rectangular channel [1-7]. The filtration flow from this channel is approximately equal to the filtration flow from a rectangular channel. The problems of filtration without backwater from channels and stream beds of arbitrary cross section in homogeneous and inhomogeneous soil, with or without a drainage layer of soil, are solved similarly by the method of successive conformal mappings.

A half-plane with a cut hole (representing the cross section of a channel or stream bed; fig. 1, a), containing the filtration area D, is mapped using the method of successive conformal mappings onto a half-plane ω with a cut out rectangle (fig. 1, c).

This mapping is done in two steps. First, the half-plane z with a cut hole representing the cross section of a channel or stream bed is mapped onto the auxiliary half-plane ζ (fig. 1b), and then ζ is mapped onto the half-plane w with a cut-out rectangle.

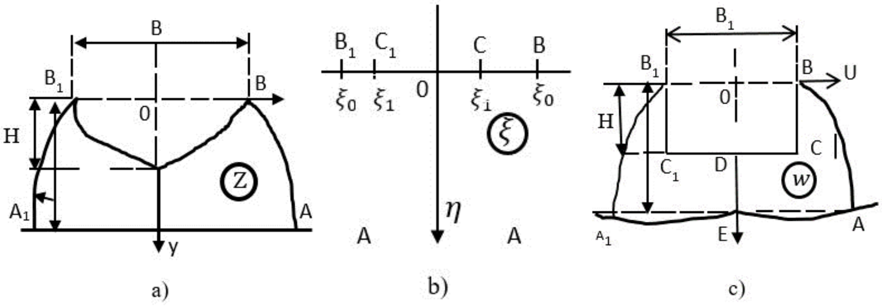

Fig. 1. *Cross section of a channel or stream bed*

The first mapping is performed by the method of successive conformal mappings using the function (for $\xi \to \zeta$)

$$\zeta_{n+1} = \frac{a_n z_n - b_n\sqrt{z_n^2 + b_n^2 - a_n^2}}{a_n - b_n} \quad (1)$$

and the second one using the function

$$w = \xi_0[E(\varphi,k) - k'^2 F(\varphi,k)] + iH \quad (2)$$

For each of these mappings, the expansion coefficient at infinity is assumed to be equal to unity, i.e., the mapping functions are normalized by the conditions

$$\left.\frac{d_\zeta}{dz}\right|_{z\to\infty} = 1;\quad \left.\frac{dw}{d_j^5}\right|_{\zeta\to\infty} = 1$$

It is easy to verify that functions (1) and (2) satisfy these conditions.

The parameter k of the elliptic integrals $E(\varphi,k)$ and $F(\varphi,k)$ in (2) is determined from the equation

$$E^{`} - k^2 K^{`} - \frac{H}{\xi_0} = 0 \quad (3)$$

where $E^{`}$ and $K^{`}$ are complete elliptic integrals modulo

$k' = \sqrt{1-k'^2}$.

Equation (3) in its final form with respect to the unknown parameter k is not solved. Therefore, approximate methods are used to determine k.

The first approximation can be determined by one of the formulas

$k_1^2 = 1 - \frac{4H}{\pi\xi_0}$ for $\frac{H}{\xi_0} \le 0{,}55$;

$k_1^2 = +\sqrt{ln^2 4 + 2\left(1 - \frac{H}{\xi_0}\right)} - ln4$ for $\frac{H}{\xi_0} > 0{,}55$

The following approximations can be found by the formula

$$k_{n+1}^2 = k_n^2 - \frac{f(k_n)}{f^{`}(k_n)}$$

where

$f(k_n) = E^{`}(k_n^{`}) - k_n^2 K^{`}(k_n^{`}) - \frac{H}{\xi_0}$.

In practice, in most cases, it is enough to find the second approximation for k'' (the error in this case does not exceed 0.6%). The depth of a rectangular channel in the half-plane w is taken equal to H (i.e., the depth of his channel in the half-plane z). The width B_1 of the channel in half-plane w is calculated by the formula

$B_1 = 2\,\xi_0(E - k^{`2} K)$,

where K and E are complete elliptic integrals of the first and second kind modulo k.

Then the problem of filtering from a rectangular channel in the half-plane w is solved.

The theory of filtration from channels of rectangular cross section is described in [6]. We present here only the final results.

a) Case T < +∞

The characteristic function of the flow has the form

$$z = -\frac{i\lambda Q_s}{4\pi K}\int_0^\zeta \frac{\Phi d\zeta}{\zeta(1-\lambda^2\zeta)(1-k^2\lambda^2\zeta)} + \frac{B}{2}$$

where $z = x + iy$ is the complex coordinate of a point in the flow area; $w = \varphi + i\psi$ is a complex potential; ψ is the filtration coefficient of the soil in which filtration occurs; Q is filtration flow per unit length of the channel; $Q_s = \frac{Q}{\psi}$ is reduced consumption; B is the channel width; K is a complete elliptic integral of the 1st kind modulo k;

$$K = \int_0^1 \frac{d\zeta}{\sqrt{\zeta(1-\zeta)}} = 2Arch\frac{1}{\sqrt{\zeta}};$$

ζ is auxiliary variable associated with the variable ω by the dependence

$$\zeta = \frac{1}{\lambda^2}\mathrm{sh}^2\frac{2iKw}{Q}$$

$k = \sin\theta$, $\lambda = \sin\vartheta$ are constant parameters that are determined from the system of equations

$$\begin{cases}\lambda'(Kg_1 - g_2) = \beta f\\ 2Tf = \pi HK'\end{cases} \quad (4)$$

H is water depth in the channel; T is the depth of the drainage layer of soil; K' is a complete elliptic integral of the 1st kind modulo $k' = +\sqrt{1-k^2}$

$$\beta = \frac{B}{2H};\ \lambda' = +\sqrt{1-\lambda^2};\ g_1 = \frac{\pi - 2\vartheta}{2\lambda'};$$

$$\vartheta = \arcsin\lambda;\ g_2 = \int_0^1 \frac{F(\sigma,k)}{\sqrt{1-\lambda'^2\xi^2}}d\xi$$

$$\sigma = \arcsin\frac{\lambda}{\sqrt{1-\lambda'^2\xi^2}};$$

$$f = \frac{\pi}{2}\int_0^1 \frac{F\left[\arcsin\left(\lambda\sin\frac{\pi}{2}\xi\right),k\right]}{\sin\frac{\pi}{2}\xi}d\xi.$$

The depression curve equation is written as

$$\pi\left(x - \frac{B}{2}\right) = 2y\mathrm{Arsh}(\lambda cs'\eta) + \frac{Q_s}{2K}\int_0^{\frac{1}{\lambda^2}sc'\eta}\frac{F(\tau,k')}{t\sqrt{1+t}}dt,$$

where $F(\tau,k')$ is an incomplete elliptic integral of the first kind modulo $k' = \pm\sqrt{1-k^2}$; $\tau = \mathrm{arctg}\,\lambda\sqrt{t}$; $cs'\eta = \frac{cn'\eta}{sn'\eta}$, $sc'\eta = \frac{sn'\eta}{cn'\eta}$ are Jacobi elliptic functions; $\eta = \frac{2Ky}{Q_s}$.

If the reduced flow rate through the slopes is denoted by Q_s' $\left(Q_s' = \frac{Q'}{x}\right)$, and the distance between the points of intersection of the depression curve with the dividing line of soils (which we consider horizontal) is $2a$, then the quantities Q_s, Q_s' and a are determined by the formulas

$$Q_s = 2T\frac{K}{K'} = \pi H\frac{g_2 + \frac{\beta}{\lambda'}f}{g_1 f}; \quad (5)$$

$$Q_s' = Q_s\frac{F(\vartheta)}{K} = 2T\frac{F(\vartheta)}{K'};$$

$$a = \frac{Q_s}{2}\left(1 - \frac{2k\lambda}{\pi K}g_s\right) \quad (6)$$

where

$$g_s = \int_0^1 \frac{K - F(\arcsin\xi,k)}{\sqrt{1-k^2\lambda^2\xi^2}}d\xi$$

b) Case $T \to +\infty$.

If $T \to +\infty$, then $k = 0, K = \frac{\pi}{2}$ and $F(\vartheta) = \vartheta$.

In this case, the characteristic function of the flow

$$z = -\frac{i\lambda Q_s}{2\pi^2}\int_0^\zeta \frac{\Phi d\zeta}{\sqrt{\zeta(1-\lambda^2\zeta)}} + \frac{B}{2}$$

depression curve equation

$$\pi\left(x - \frac{B}{2}\right) = 2y\mathrm{Arsh}\frac{\lambda}{\mathrm{sh}\frac{\pi y}{Q_s}} + \int_0^{\frac{1}{\lambda^2}\mathrm{sh}^2\frac{\pi y}{Q_s}}\frac{\mathrm{Arsh}\,\lambda\sqrt{t}}{t\sqrt{1+t}}dt;$$

reduced flow

$$Q_s = \frac{\pi H}{\mathcal{J}} \quad (7)$$

or

$$Q_s = \frac{\pi^2 B}{2J}$$

where

$$\mathcal{J} = \int_0^1 \frac{\arcsin \lambda\sqrt{t}}{t\sqrt{1-t}} dt, \; J = \int_1^{\frac{1}{\lambda^2}} \frac{\arccos \lambda\sqrt{t}}{t\sqrt{t-1}} dt$$

slope flow

$$Q_s' = Q_s \frac{2\vartheta}{\pi}$$

$$a = \frac{Q_s}{2}.$$

The parameter λ is determined from the equation $J - \beta\mathcal{J} = 0$.

In the case of $T \to +\infty$, nomograms [6] were constructed to determine the values of λ and $\frac{Q_s}{H}$ (fig. 2).

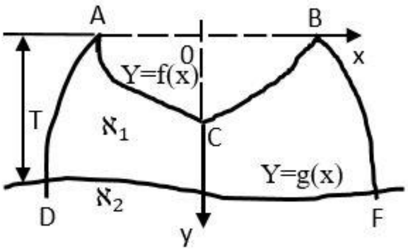

Fig. 2. *Design scheme of the channel cross section*

Analytical solutions and computational algorithms built on their basis for determining exact or approximate values of the main filtration characteristics during filtration from channels and stream beds in the case of a finite depth of the drainage layer of soil ($T \ll +\infty$) are much more complicated (require much more computational work) than in case of infinite depth of the drainage layer ($T \to +\infty$).

Therefore, when solving problems of filtration without backwater from channels and stream beds, in the case of a sufficiently large depth T, the occurrence of the drainage layer of soil is taken approximately $T \to +\infty$, given that the errors in determining the main filtration characteristics from such a replacement are insignificant, and the solution of the problem is greatly simplified.

Quite naturally, the question arises, what should be the depth T of the drainage layer of soil, so that when solving the problems of filtration from a channel or stream bed, one could take T→+∞ and, at the same time, the errors in determining the main filtration characteristics do not exceed the allowable.

In the general formulation, this problem regarding the flow rate for filtration can be formulated as follows.

For a channel or stream bed of arbitrary cross section, the cross-sectional shape of the channel is known, i.e., the equation $y = f(x)$ is given for the line representing the perimeter of the cross-section of the channel or stream bed, and the dividing line of the filtration area (the soil in which filtration occurs) and the drainage soil layer (having a much larger coefficient filtration than the filtration coefficient of the soil in which filtration occurs), i.e., given the equation $y = g(x)$ of this line.

If the soil in which filtration occurs (area D_1) and the drainage layer (area D_2) are heterogeneous, and their filtration coefficients are respectively equal to $\aleph_1(x, y)$ and $\aleph_2(x, y)$, then the following condition is assumed to be fulfilled:

$$\max_{(x,y)\in D_1} \aleph_1(x, y) \ll \min_{(x,y)\in D_1} \aleph_2(x, y)$$

The smallest distance from the line of the water mirror in the channel or stream bed to the drainage layer of soil will be denoted by T (fig. 2).

The task is:

1) To find the error that is obtained when determining the flow rate for filtration Q from a channel or stream bed, if we assume that the depth of the drainage layer is equal to infinity ($T = +\infty$) instead of the finite depth of its occurrence;

2) To determine at what minimum value of T the deviation of the filtration flow Q_T at $T < +\infty$ from the filtration flow Q_∞ at $T = +\infty$ will not exceed the specified number $\varepsilon(\%)$.

In practice, the problem can be formulated as follows: at what depth T of the drainage layer of the soil, the flow rate for filtration does not depend on T and the problem of calculating the filtration can be solved by considering $T = +\infty$. In such a general formulation, it is currently not possible to obtain an analytical solution of the problem in a closed finite form. However, having solved the problem in a particular case, the results can be used in other cases.

Let us find a solution to the problem for the case of filtration from a channel of rectangular cross section, since for such a channel there are analytical solutions to the filtration problem, and these solutions are simpler than the solutions of the corresponding problems for other channels.

Let us consider this problem under the conditions indicated above, i.e., we assume that the liquid is incompressible, its motion in the soil is stationary and obeys the Darcy law. The forces of inertia and the phenomenon of capillary rise of water (liquid) in the soil are not taken into account. It is also assumed that evaporation and infiltration are absent.

To solve the problem, we use the values of the reduced filtration flow from channels of rectangular cross section at $T < +\infty$ and $T = +\infty$, as well as, the dependence of $\frac{B}{H}$ (upper numbers) and $\frac{T}{H}$ (lower numbers) on the auxiliary parameters k and λ. Also shows the values of $\frac{a}{H}$ (lower numbers) so that you can see how different the values of $\frac{a}{H}$ are from $\frac{Q_s}{2H}$ at $T < +\infty$.

On the basis of the above formulas

$$\frac{B}{H} = \frac{4 \arccos \lambda}{\pi f_1}(K - g); \quad (8)$$

$$\frac{T}{H} = \frac{K'}{f_1}, \quad (9)$$

where $f_1 = \frac{2}{\pi} f$; $g = \int_0^1 F(\sigma, k) d\xi$;

$$\sigma = \arcsin \frac{\lambda}{\sin(\arcsin \lambda + \xi \arccos \lambda)}.$$

Using the formula

$$\frac{Q_s}{H} = \frac{2K}{f_1} \quad (10)$$

we get

$$\frac{Q}{H} = \frac{Q_s}{H}\left(0{,}5 - \frac{k\lambda}{\pi K} g_3\right) \quad (11)$$

Formulas (8)-(11) are obtained by elementary transformations from formulas (9)-(6).

In this case, to calculate the values of $\frac{Q_s}{H}$, the formula (obtained from (7)) [6] was used

$$\frac{Q_s}{H} = \frac{\pi}{\tilde{\gamma}_1}, \qquad (12)$$

where $\tilde{\gamma}_1 = \int_0^1 \frac{\arcsin\left(\lambda \sin\frac{\pi}{2}\xi\right)}{\sin\frac{\pi}{2}\xi} d\xi$.

Formula (12) is also obtained from formula (10) for $k = 0$ and $K = \frac{\pi}{2}$. Let us introduce the notation:

$\left(\frac{Q_s}{H}\right)_T$ is the value of $\frac{Q_s}{H}$ provided that T<+∞;

$\left(\frac{Q_s}{H}\right)_\infty$ is the value of $\frac{Q_s}{H}$ provided that $T = +\infty$ (for the same channel);

$$\delta = \frac{\left(\frac{Q_s}{H}\right)_T - \left(\frac{Q_s}{H}\right)_\infty}{\left(\frac{Q_s}{H}\right)_T} 100\%.$$

Based on the given values of $\frac{B}{H}$ and $\frac{T}{H}$, it is possible to approximately, with sufficient accuracy for practice, determine the error that is obtained when calculating the reduced filtration flow from a rectangular channel in the case of replacing $\frac{T}{H} < +\infty$ with $\frac{T}{H} = +\infty$.

Since the obtained results do not depend on the filtration coefficient ℵ, they remain valid for the case of filtration in soils with arbitrary (variable) filtration coefficients, in particular, in multilayer soils.

The results obtained can also be used in the calculation of filtration from channels and stream beds of arbitrary cross section. To do this, it suffices to describe a rectangle around the cross section of the channel or stream bed and find the value δ for a rectangular channel with such a cross section.

In this case, the problem was considered under the condition that the dividing line of soils - the areas of filtration and the drainage layer is a straight horizontal line.

If the soil dividing line is arbitrary, then this line can be taken as a horizontal straight line passing through the upper point of the true soil dividing line and the considered method can be used to estimate the error. Thus, the result obtained can, generally speaking, be used in solving the problem in the general case.

If the resulting error δ in determining the filtration flow in the case of replacing $T < +\infty$ with $T = +\infty$ exceeds the practically specified one, then to determine it, you can use the formula

$$Q_T = \frac{100}{100-\delta} Q_\infty \qquad (13)$$

where Q_T is the seepage flow at $T < +\infty$; Q_∞ is seepage flow at $T = +\infty$ (for the same channel); δ is error in % of Q_T, which is obtained by replacing $T < +\infty$ with $T = = +\infty$; it is easy to find it from the nomogram given here (if it does not exceed 30%).

Tables or nomograms can be compiled to determine seepage flow from some channels and stream beds at $T = +\infty$. To determine the seepage flow from a channel of rectangular cross section, formula (13) is proposed; it is easy to determine the seepage flow from channels and stream beds at any depth of the drainage soil layer.

The choice of the cross-sectional shape of the channel in the design of its construction plays an important role. One of the main operational indicators of channels is the filtration flow, which depends on the shape of the channel's cross section for a given area. For most channels, an increase in filtration flow is a negative phenomenon. Therefore, it is desirable that it (under certain given conditions) be minimal.

In the general formulation, the problem can be formulated as follows. Given a channel of arbitrary cross section. Filtration without backwater from the channel takes place, provided that $T \leqslant +\infty$ (T is the depth of the drainage soil layer).

Let $y = g(x)$ be the equation of the dividing line of soils, and $g(x)$ be a known given function; $y = f(x)$ be the equation of the bottom of the channel cross section, and $f(x)$ be an unknown function; S be the cross-sectional area of the channel (constant given value).

It is required to find such a function $f(x)$ that satisfies the condition $\int_{x_1}^{x_2} f(x)dx = S$, so that the seepage flow rate $Q = Q[f(x)]$ is minimal.

In such a general setting, it is currently impossible to obtain an analytical solution to this problem, since the functional Q [f (x)] is unknown. There are solutions to this problem in several special cases.

Conclusion

The filtration flow from a channel with a given cross-sectional area at an infinite depth of the drainage layer will be the smallest if the channel cross-section has the shape of a semicircle.

References:

1. Studies of filtration and evaporation from the reservoir and compilation of water balance / Az. Research Institute of Water Problems. – Baku – 1988 (in Russian).
2. Ivanenko Yu.G. Theoretical principles and solutions of special problems of hydraulics of open stream beds / Ivanenko Yu.G., Tkachev A.A. – Novocherkassk: “Lik”, NGMA, 2013, 203 p. (in Russian).
3. Imanov F.A., Mammadov X.B., Rajabov R.F. The current state of water resources uses in the Kur basin countries. // Global and regional problems of extreme hydrometeorological events. Materials of the scientific conference dedicated to the 70th anniversary of prof. M.A. Mammadov's birth. Baku, 2008, pp. 94-104. (in Azerbaijani).
4. Ways to increase the efficiency of irrigated agriculture. Scientific and practical journal Issue No. 2(70), 2018 (in Russian).
5. Filchakov P.F., Rudchenko P.A., Yaremchuk F.P. Dependence of filtration flow from channels and stream beds on the depth of the drainage layer - In the book: Mathematical modeling on continuous and discrete media. K., 1974, pp. 98-105 (in Russian).
6. Rudchenko P.A. On the issue of filtration from a rectangular channel with an infinite depth of the drainage layer. - In the book: Boundary Value Problems in Filtration Theory. K.,1973, pp. 124-125 (in Russian).
7. Rudchenko P.A. To the calculation of filtration from channels and stream beds. Applied mechanics, 1966, 11, issue 2, pp. 117-125 (in Russian).