MATHEMATICAL SCIENCES

EXPANSİON OF THE GREEN FUNCTİON OF A SECOND ORDER LİNEAR İNTEGRO-DİFFERENTİAL OPERATOR ON THE WHOLE AXİS

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Abstract

Let us consider the following integro-differential equation

$$Ly = -y''(x) + p(x)y'(x) + \int_{-\infty}^{\infty} k(x,s,\rho)y'(s)ds - \lambda y(x), \tag{3.1}$$

where $R_{\infty}^{+} = (-\infty, \infty)$, $\lambda = \rho^2$, $Jm\rho > 0$.

Let $\rho(x)$ be a complex valued summable function in the interval $R_{-}^{+} = (-\infty, \infty)$ and the kernel of the integro-differential equation (3.1) have the form:

$$K(x,s,\rho) = \begin{cases} e^{-\rho\omega_1(s-x)}\rho(s)k^*(x,s,\rho) & for \quad s < x, \\ e^{-\rho\omega_2(s-x)}\rho(s)k^*(x,s,\rho) & for \quad s > x, \end{cases} \tag{3.2}$$

where ω_1, ω_2 are various roots of second degree from -1, while $k^*(x,s,\rho)$ is a complex valued continous bounded function in totality of variables (x,s,ρ), $-\infty < x, s < \infty$, $|Jm\rho| < \varepsilon_0/2$ and analytic with respect to ρ in the domain $|Jm\rho| < \varepsilon_0/2$ for each fixed (x,s), $-\infty < x, s < \infty$.

Furthermore, let the following condition be fulfilled:

$$|\rho(x)|\left[1 + \int_{-\infty}^{+\infty} |k^*(s,x,\rho)|ds\right] < ce^{-2\varepsilon_0|x|} \tag{3.3}$$

where $\varepsilon_0 > 0$ and

$$\int_{-\infty}^{\infty} e^{2\varepsilon_0|x|}|\rho(x)|\left[1 + \int_{-\infty}^{\infty} |k^*(s,x,\rho)|ds\right]dx < B_1 < \infty \tag{3.4}$$

where B_1 is independent of ρ.

We choose the contour $C_{R,\delta}$ on the ρ plane of the following form: $C_{R,\delta}$ consists of the straightline $\rho_2 = \delta > 0$ from $\rho_2 = -(R^2 - \delta^2)^{\frac{1}{2}}$ to $\rho_1 = (R^2 - \delta^2)^{\frac{1}{2}}$ and the arch of the circle $\rho = \mathrm{Re}^{w_1\varphi}$ from $\varphi = \arcsin \delta/R$ to $\varphi = \pi - \arcsin\dfrac{\delta}{R}$.

Choose δ such small and R_0 such large that $\sqrt{\lambda}$ lies interior to the contour $C_{R,\delta}$ for $R \geq R_0$ and so that all $\sqrt{\lambda}$ where λ is an eigen value and $Jm\sqrt{\lambda} > \delta$ lie interior to $C_{R,\delta}$.

Then we consider the contour integral

$$J(R,\delta) = \oint_{C_{R,\delta}} \frac{T(x,\tau,\rho)}{\rho^2 - \lambda}\rho d\rho \text{ for } R \geq R_0.$$

The point $\sqrt{\lambda}$ lies interior to the contour $C_{R,\delta}$. Now apply to $J(R,\delta)$ the residue theorme

$$T(x,\tau,\rho)=M_1(x,\tau,\rho)+\int_{-\infty}^{\infty}\frac{D(x,\tau,\rho)}{D(\rho)}M_1(t,\tau,\rho)dt,$$

where

$$D(x,t,\rho)=N_1(x,t,\rho)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}D_n(x,t,\rho).$$

$T(x,\tau,\rho)$ is the ratio of two functions each of which is analytic for $Jm\rho>0$. Therefore, $\rho^2=\lambda_1,\lambda_2,\ldots$ located interior to $C_{R,\delta}$ will be poles.

Then we can write the expansion $T(x,\tau,\rho)$ in the vicinity of the point λ_i:

$$T(x,\tau,\rho)=\sum_{p=1}^{m_i}\frac{T_\rho^{(i)}(x,r)}{\left(\rho^2-\lambda_i\right)^\rho}+F(x,\tau,\rho).$$

It is proved that if (3.3) is fulfilled, $D(\rho)$ does not vanish at the points of the real axis ρ_1 and the zeros $D(\rho)$ are simple and $T(x,\tau,\rho,\lambda)$ is a Green function of the integro-differential operator L, then for any point λ, not belonging to the spectrum of the integro-differential operator L

$$T(x,\tau,\lambda)=\sum_{i=1}^{n}\frac{y_i(x)\overline{y_i(r)}}{(\lambda-\lambda_i)(y_i,y_i)}+\frac{1}{2\pi}\int_0^\infty\frac{\sum_{i=1}^{2}Ay_i(x,\rho)y_{3-i}(\tau,\rho_1)}{\rho_1^2-\lambda}d\rho_1$$

where the integral from the right converges absolutely and uniformly with respect to x,τ, in the interval $-\infty<x,\ \tau<\infty$

Keywords: integro-differential, expansion, Green function, eigen function, values, equation, variable, integral teorem, kernel, whole axis complex-valued, let converge, spectrum, point, contour.

I. Introduction. Let us consider the integro-differential equation

$$Ly=-y''(x)+p(x)y'(x)+\int_{-\infty}^{\infty}k(x,s,\rho)y'(s)ds-\lambda y(x),$$

and choose the contour $C_{R,\delta}$ on the ρ plane of the following form:

$C_{R,\delta}$ consists of the straightline $\rho_2=\delta>0$ from $\rho_2=-\left(R^2-\delta^2\right)^{\frac{1}{2}}$ to $\rho_1=\left(R^2-\delta^2\right)^{\frac{1}{2}}$ and the arch of the circle $\rho=\mathrm{Re}^{w_1\varphi}$ from $\varphi=\arcsin\frac{\delta}{R}$ to $\varphi=\pi-\arcsin\frac{\delta}{R}$.

Choose δ such small and R_0 such large that $\sqrt{\lambda}$ lies interior to the contour $C_{R,\delta}$ for $R\geq R_0$ and so that $\sqrt{\lambda}$ lies interior to the contour $C_{R,\delta}$ for $R\geq R_0$ and so that all $\sqrt{\lambda}$ where λ is an eigen-value of the integro-differential operator L and $Jm\sqrt{\lambda}>\delta$ lie interior to $C_{R,\delta}$.

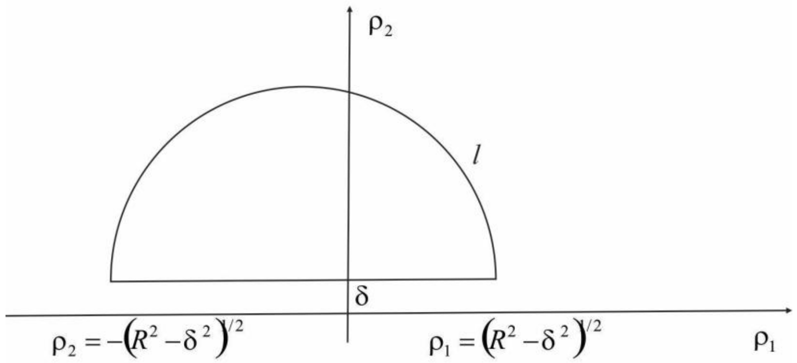

Fig.

Let us consider the contour integral

$$J(R,\delta)=\oint_{C_{R,\delta}}\frac{T(x,\tau,\rho)}{\rho^2-\lambda}\rho d\rho \text{ for } R\geq R_0$$

the point $\sqrt{\lambda}$ lies interior to the contour $C_{R,\delta}$. The singularity of the integrand function will be at the point $\rho=\sqrt{\lambda}$ and at the points $\sqrt{\lambda_k}\left(k=\overline{1,n}\right)$, where λ_k are singuliarities of the functions

$$T(x,\tau,\rho)=M_1(x,\tau,\rho)+\int_{-\infty}^{\infty}\Gamma_1(x,t,\rho)\mu_1(t,\tau,\rho)dt=M_1(x,\tau,\rho)+$$

$$+\int_{-\infty}^{\infty}\frac{D(x,t,\rho)}{D(\rho)}m_1(t,\tau,\rho)dt,$$

where

$$D(x,t,\rho)=N_1(x,t,\rho)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}D_n(x,t,\rho),$$

$$D_n(x,t,\rho)=\frac{1}{(2\rho\omega_1)n+1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}N_1\rho\begin{pmatrix}x,t_1,t_2,\dots,t_n\\ t,t_1,t_2,\dots,t_n\end{pmatrix}\prod_{k=1}^{n}\rho(t_k)dt_k,$$

$$N_1\rho\begin{pmatrix}x,t_1,t_2,\dots,t_n\\ t,t_1,t_2,\dots,t_n\end{pmatrix}=\begin{vmatrix}N_1p(x,t) & N_1p(x,t_1), & \cdots, & N_1p(x,t_n)\\ N_1(t_1,t) & N_1p(t_1,t_1), & \cdots, & N_1p(t_1,t_n)\\ \cdots & \cdots & \cdots & \cdots\\ N_1(t_n,t) & N_1p(t_n,t_1), & \cdots & N_1p(t_n,t_n)\end{vmatrix}$$

Repeating the proof of theorem 2.1 we can easily prove that $D(x,t,\rho)$ allows analytic continuation through the positive semi-axis in the domain E.

Obviously, $T(x,\tau,\rho)$ is the ratio of two functions one of shich is analytic for $Jm\rho>0$.

Therefore, $\rho^2=\lambda_1,\lambda_2,\dots$ located interior to $C_{R,\delta}$ will be poles.

Then we can write the expansion $T(x,\tau,\rho)$ in the vicinity of the point λ_i:

$$T(x,\tau,\rho)=\sum_{\rho=1}^{m_i}\frac{T_\rho^{(i)}(x,\tau)}{\left(\rho^2-\lambda_i\right)^\rho}+F(x,\tau,\rho),$$

where $F(x,\tau,\rho)$ is a regular function in the vicinity of the point $\rho=\sqrt{\lambda_i}$, while m_i is an order of poles and the residual of the function $T(x,\tau,\rho)$ with respect to $\rho^2=\lambda_i$ equals $\frac{1}{2}T(x,\tau,\lambda)$

$$-\frac{1}{2}\sum_{\rho=1}^{m_i}\frac{T_\rho^{(i)}(x,\tau)}{(\lambda-\lambda_i)^\rho}.$$

The residual with respect to M_i will be $\frac{1}{2}T(x,\tau,\lambda)$.

Therefore

$$J(R,\rho)=\pi\omega_1T(x,\tau,\lambda)-\pi\omega_1\sum_{i=1}^{n(\delta)}\sum_{\rho=1}^{m_2}\frac{T_\rho^{(i)}(x,\tau)}{(\lambda-\lambda_i)^\rho},$$

where $n(\delta)$ denotes the number of eigen-values lying interior to $C_{R,\delta}$.

Hence

$$T(x,\tau,\lambda)=\sum_{i=1}^{n(\delta)}\sum_{\rho=1}^{m_i}\frac{T_{\rho}^{(i)}(x,\tau)}{(\lambda-\lambda_i)^{\rho}}+\frac{1}{\pi\omega_1}J(R,\delta). \tag{3.5}$$

We now calculate $J(R,\delta)$

$$J(R,\delta)=\int\limits_{\arcsin\frac{\delta}{R}}^{\pi-\arcsin\delta/R}\frac{T\left(x,\tau,\mathrm{R}e^{w,\varphi}\right)\omega_1R^2e^{2w_1\varphi}}{R^2e^{2w_1\varphi}-\lambda}d\varphi+$$

$$+\int\limits_{-\left(R^2-\delta^2\right)^{\frac{1}{2}}}^{\left(R^2-\delta^2\right)^{\frac{1}{2}}}\frac{T(x,\tau,\rho_1+\omega_1\delta)(\rho_1+\omega_1\delta)}{(\rho_1+\omega_1\delta)^2-\lambda}d\rho_1 \tag{3.6}$$

Prove that

$$\lim_{R\to\infty}\int\limits_{\arcsin\delta/R}^{\pi-\arcsin\delta/R}\frac{T\left(x,\tau,\mathrm{R}e^{w_1,\varphi}\right)\omega_1R^2e^{2w_1\varphi}}{R^2e^{2w_1\varphi}-\lambda}d\varphi=0.$$

We know that

$$T\left(x,\tau,\mathrm{R}e^{w_1,\varphi}\right)=\frac{T_1^{(i)}(x,\tau)}{R^2e^{2w_1\varphi}-\lambda_i}+\frac{T_2^{(i)}(x,\tau)}{\left(R^2e^{2w_1\varphi}-\lambda_i\right)^2}+\cdots+\frac{T_m^{(i)}(x,\tau)}{\left(R^2e^{2w_1\varphi}-\lambda_i\right)^{m_i}}. \tag{3.7}$$

Therefore based on (3.7) at all the points of the semi-cirle e we have:

$$\left|T\left(x,\tau,\mathrm{R}e^{w_1,\varphi}\right)\right|\le\frac{\left|T_1^{(i)}(x,\tau)\right|}{R^2}+\frac{1}{R^2}\left[\frac{\left|T_2^{(i)}(x,\tau)\right|}{R^2}+\cdots+\frac{\left|T_m^{(i)}(x,\tau)\right|}{R^2}\right] \tag{3.8}$$

The expression in the paranthesis of (3.8) can be made arbitrarily small, for example less than a unit, for fixed x,τ beginning from a rather large R.

Consequently, from the inequality (3.8) we obtain:

$$\left|T\left(x,\tau,\mathrm{R}e^{w_1,\varphi}\right)\right|\le\frac{\left|T_1^{(i)}(x,\tau)\right|+1}{\rho^2} \tag{3.9}$$

everywhere on the semi-circle l, beginning with a rather large R.
Using the inequality (3.9) we obtain:

$$\left|\int\limits_{\arcsin\delta/R}^{\pi-\arcsin\delta/R}\frac{T\left(x,\tau,\mathrm{R}e^{w_1,\varphi}\right)\omega_1R^2e^{2w_1\varphi}}{R^2e^{2w_1\varphi}-\lambda}d\varphi\right|\le\frac{\left|T_1^{(i)}(x,\tau)\right|+1}{R^2}\pi,$$

i.e. for fixed x,τ we have:

$$\lim_{R\to\infty}\int\limits_{\arcsin\delta/R}^{\pi-\arcsin\delta/R}\frac{T\left(x,\tau,\mathrm{R}e^{w_1,\varphi}\right)\omega_1R^2e^{2w_1\varphi}}{R^2e^{2w_1\varphi}-\lambda}d\varphi=0.$$

We can also easily prove that

$$\left|\frac{(\rho_1+\omega_1\delta)T(x,\tau,\rho_1+\omega_1\delta)}{(\rho_1+\omega_1\delta)^2-\lambda}\right|\le\frac{c}{\rho_1^2}$$

for a rather large R.
The obtained estimates imply that

$$\lim_{R\to\infty}J(R,\delta)=\int\limits_{-\infty}^{\infty}\frac{(\rho_1+\omega_1\delta)T(x,\tau,\rho_1+\omega_1\delta)}{(\rho_1+\omega_1\delta)^2-\lambda}d\rho_1$$

exists and the integral from the right converges absolutely and uniformly with respect to x,τ for $Jm\rho\ge\delta_1>\delta>0$.

So, having noted it, from formula (3.5) we pass to the limit as $R \to \infty$ and obtain:

Theorem 3.1. For $\delta > 0$ for the Green function $T(x, \tau, \lambda)$ we have the expansion

$$T(x,\tau,\lambda)=\sum_{i=1}^{n(\delta)}\sum_{\rho=1}^{m_i}\frac{T_\rho^{(i)}(x,\tau)}{(\lambda-\lambda_i)^\rho}-\frac{\omega_1}{\pi}\int_{-\infty}^{\infty}\frac{(\rho_1+\omega_1\delta)T(x,\tau,\rho_1+\omega_1\delta)}{(\rho_1+\omega_1\delta)^2-\lambda}d\rho_1 \tag{3.10}$$

In what follows, we will assume that:

1) The condition (3.3) is fulfilled.
2) The function $D(\rho)$ does not vanish at the points of the real axis ρ_1
3) The zeros of $D(\rho)$ are simple.

If (3.3) is fulfilled, then by virtue of theorem 2.1, $D(\rho)$ is analytic in the domain E.

Furthermore, by virtue of theorem 2.1 $D(\rho)$ for rather large ρ does not vanish in the domain E, therefore the number of zeros of $D(\rho)$ may only be finite.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be all eigen-values, while $y_1(x), y_2(x), \ldots y_n(x)$ be appropriate eigen-values of the integro-differnetial operator L.

Since $\lambda = 0$ (theorem 2.2) is not an eigen-value and $D(\rho)$ does not vanish at the points of the real axis ρ_1, then the integral (3.10) converges for $\delta \to 0$ as well.

If the zeros of $D(\rho)$ are simple, then

$$T_\rho^{(i)}(x,\tau)=T^{(i)}(x,\tau)=-\frac{y_i(x)\overline{y_i(\tau)}}{(y_i,y_i)},$$

where

$$(y_i,y_i)=\int_{-\infty}^{\infty}y_i(\tau)\overline{y_i(\tau)}d\tau.$$

Then by virtio of finiteness of the number of eigen values and the condition 2) the equality (3.10) as for $\delta \to 0$ takes the form:

$$T(x,\tau,\lambda)=\sum_{i=1}^{n}\frac{y_i(x)\overline{y_i(\tau)}}{(\lambda-\lambda_i)(y_i,y_i)}-\frac{\omega_1}{\pi}\int_{-\infty}^{\infty}\frac{\rho_1T(x,\tau,\rho_1)}{\rho_1^2-\lambda}d\rho_1=$$

$$=\sum_{i=1}^{n}\frac{y_i(x)\overline{y_i(\tau)}}{(\lambda-\lambda_i)(y_i,y_i)}-\frac{\omega_1}{\pi}\int_{0}^{\infty}\frac{\rho_1T(x,\tau,\rho_1)-\rho_1T(x,\tau,-\rho_1)}{\rho_1^2-\lambda}d\rho_1 \tag{3.11}$$

We now prove the following theorem.

Theorem 3.2. We have

$$T(x,\tau,\rho_1)-T(x,\tau,-\rho_1)=-\frac{1}{2\rho_1\omega_1}\sum_{i=1}^{2}Ay_i(x,\rho_1)\cdot y_{3-i}(\tau,\rho_1)$$

where

$$Ay_i(x,\rho_1)=y_i(x,\rho_1)+\int_{-\infty}^{\infty}\frac{D\left(x,t,(-1)^{3-i}\rho_i\right)}{D\left((-1)^{3-i}\rho_i\right)}y_i(t,\rho_1)dt.$$

Proof. From

$$T(x,\tau,\rho)=M_1(x,\tau,\rho)+\int_{-\infty}^{\infty}\frac{D(x,t,\rho)}{D(\rho)}M_1(t,\tau,\rho)dt$$

it is clear that

$$T(x,\tau,\rho_1)=\begin{cases}-\dfrac{y_1(x,\rho_1)y_2(\tau,\rho_1)}{2\rho_1\omega_1}-\dfrac{y_2(\tau,\rho_1)}{2\rho_1\omega_1}\int\limits_{-\infty}^{\infty}\dfrac{D(x,t,\rho_1)}{D(\rho_1)}y_1(t,\rho_1)dt \;\; for\; \tau<x,\\ -\dfrac{y_2(x,\rho_1)y_r(\tau,\rho_1)}{2\rho_1\omega_1}-\dfrac{y_1(\tau,\rho_1)}{2\rho_1\omega_1}\int\limits_{-\infty}^{\infty}\dfrac{D(x,t,\rho_1)}{D(\rho_1)}y_2(t,\rho_1)dt \;\; for\; \tau>x,\end{cases} \tag{3.12}$$

$$T(x,\tau,-\rho_1)=\begin{cases}\dfrac{y_1(x,-\rho_1)y_2(\tau,-\rho_1)}{2\rho_1\omega_1}+\dfrac{y_2(\tau,-\rho_1)}{2\rho_1\omega_1}\int\limits_{-\infty}^{\infty}\dfrac{D(x,t,-\rho_1)}{D(-\rho_1)}y_1(t,-\rho_1)dt \;\; for\; \tau<x,\\ \dfrac{y_1(\tau,-\rho_1)y_2(x,-\rho_1)}{2\rho_1\omega_1}+\dfrac{y_1(\tau,-\rho_1)}{2\rho_1\omega_1}\int\limits_{-\infty}^{\infty}\dfrac{D(x,t,-\rho_1)}{D(-\rho_1)}y_2(t,-\rho_1)dt \;\; for\; \tau>x,\end{cases} \tag{3.13}$$

At first we consider the case $\tau<x$.

Obviously,

$y_i(x,-\rho_1)=y_{3-i}(x,\rho_1)$ and $y_i(\tau,-\rho_1)=y_{3-i}(\tau,\rho_1),\ i=1,2.$

Therefore

$$T(x,\tau,\rho_1)-T(x,\tau,-\rho_1)=-\frac{y_1(x,\rho_1)y_2(\tau,\rho_1)+y_2(x,\rho_1)y(\tau,\rho_1)}{2\rho_1\omega_1}-$$

$$-\frac{y_2(\tau,\rho_1)}{2\rho_1\omega_1}\int\limits_{-\infty}^{\infty}\frac{D(x,t,\rho_1)}{D(\rho_1)}y_1(t,\rho_1)dt-\frac{y_1(\tau,\rho_1)}{2\rho_1\omega_1}\int\limits_{-\infty}^{\infty}\frac{D(x,t,-\rho_1)}{D(-\rho_1)}y_2(t,\rho_1)dt=$$

$$=-\frac{\sum\limits_{i=1}^{2}y_i(x,\rho_1)y_{3-i}(\tau,\rho_1)}{2\rho_1\omega_1}-\frac{\sum\limits_{i=1}^{2}y_{3-i}(\tau,\rho_1)}{2\rho_1\omega_1}\int\limits_{-\infty}^{\infty}\frac{D\left(x,t,(-1)^{3-i}\rho_1\right)}{D\left((-1)^{3-i}\rho_1\right)}y_i(t,\rho_1)dt=$$

$$=-\frac{1}{2\rho_1\omega_1}\sum_{i=1}^{2}Ay_i(x,\rho_1)y_{3-i}(\tau,\rho_1). \tag{3.14}$$

In this way, to the same expression (3.14) we arrive for $\tau>x$ as well.

Substituting expression $T(x,\tau,\rho_1)-T(x,\tau,-\rho_1)$ in (3.11), we obtain:

$$T(x,\tau,\lambda)=\sum_{i=1}^{n}\frac{y_i(x)\overline{y_i(\tau)}}{(\lambda-\lambda_i)(y_i,y_i)}+\frac{1}{2\pi}\int\limits_{0}^{\infty}\frac{\sum\limits_{i=1}^{2}Ay_i(x,\rho_1)y_{3-i}(\tau,\rho_1)}{\rho_1^2-\lambda}d\rho_1 \tag{3.15}$$

So, we proved

Theorem 3.3. If conditions 1) 2) 3) are fulfilled, and $T(x,\tau,\lambda)$ is a Green function of the integro-differential operator L, then for any point λ not belonging to the spectrum of the integro-differential operator L

$$T(x,\tau,\lambda)=\sum_{i=1}^{n}\frac{y_i(x)\overline{y_i(\tau)}}{(\lambda-\lambda_i)(y_i,y_i)}+\frac{1}{2\pi}\int\limits_{0}^{\infty}\frac{\sum\limits_{i=1}^{2}Ay_i(x,\rho_1)y_{3-i}(\tau,\rho_1)}{\rho_1^2-\lambda}d\rho_1,$$

where the integral from the right converges absolutely and uniformly with respect to x,τ in the interval $-\infty<x,\tau<\infty$.

This formula looks simplest if the function $y_i(x),\overline{y_i(\tau)}$ is normalized so that $(y_i,y_i)=1$.

Then

$$T(x,\tau,\lambda)=\sum_{i=1}^{n}\frac{y_i(x)\overline{y_i(\tau)}}{\lambda-\lambda_i}+\frac{1}{2\pi}\int\limits_{0}^{\infty}\frac{\sum\limits_{i=1}^{2}Ay_i(x,\rho_1)y_{3-i}(\tau,\rho_1)}{\rho_1^2-\lambda}d\rho_1 \tag{3.16}$$

We now find the expansion of the given function $f(x)$ in eigen functions of the integro-differential operator L

By D we denote the totality of all functions $f(x)$ satisfying the following conditions.

1) $f(x)$ is summable in the interval R_{-}^{+}

2) $f'(x)$ exists and absolutely continuous in each finite interval $[\alpha, \beta] \subset R_{-}^{+}$, where $\alpha < \beta$,

3) $L(f) \in L^2(R_{-}^{+})$.

Let $f(x) \in D$. Then

$$f(x) = \sum_{i=1}^{n} y_i(x) \int_{-\infty}^{\infty} (f, y_i) dx + \frac{1}{2\pi} \int_{0}^{\infty} \sum_{i=1}^{2} A y_i(x, \rho_1) y_{3-i}(\tau, \rho_1) d\rho_1 \tag{3.17}$$

where

$$\psi_{3-i}(\rho_1) = \int_{-\infty}^{\infty} y_{3-i}(\tau, \rho_1) f(\tau) d\tau.$$

References:

1. Naimark M.A. Studying spectrum and expansion of a second order not self-adjoint differential operator in eigen functions on a semi-axis. Trudy Moskovskogo Matematicheskogo Obshestva, №3, 1954, p.181-270.

2. Naimark M.A. Linear differential operators. Nauka, Moscow, 1969, p.24-50.

3. Naimark M.A. Spectral analysis of not self-adjoint operators. UMN, 11, 6(72), 1956, p.183-202.

4. Sansone J. Ordinary differential equations. I-II, 1953.

5. Shabat B.V. Introduction to complex analysis. Moscow “Nauka” 1965, part II. p.291-368.

6. Almammadov M.S. Expansion of the Green function of integro-differential operator L_θ. Norwegian Journal of Development of the Intern. Science, №106/2023, p.29-33.

7. Almammadov M.S. On the spectrum of a fourth order linear integro-differential operator. Doklady AN SSSR, v.29, №3, 1988, p.525-529.