MATHEMATICAL SCIENCES

CONSTRUCTION METHOD OF AUXILIARY FUNCTION IN DIFFERENTIAL MEAN VALUE THEOREM

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Abstract

Constructing the auxiliary function is a key method in the proof problems of the differential mean value theorem.Several typical methods of constructing the auxiliary function in the proof problems of the mean value are discussed and summarized, and verified through specific examples.

Keywords: The Mean Value Theorem of Differentiation; Differential Equation; Primitive Function; Auxiliary Function

The differential mean value theorem is one of the important theorems in differential calculus. It is not only the theoretical basis for the application of derivatives but also the bridge for the application of differential calculus. Therefore, the study of the proof problems of the differential mean value theorem can not only deepen the understanding of the essence of the mean value theorem, but also improve students' mathematical thinking and ability to solve mathematical problems.

Most of the proof problems of the differential mean value theorem need to be completed with the help of the auxiliary function, and how to construct the appropriate auxiliary function becomes the key to solving such problems. This article discusses and summarizes several methods of constructing the auxiliary function, and gives detailed methods and steps through examples.

I.Preliminary Knowledge

Theorem1 [1] (Rolle's Theorem) If the function $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable within the open interval (a,b), and $f(a)=f(b)$, then there is at least one point ξ $(a<\xi<b)$,such that the equation $f'(\xi)=\mathbf{0}$ holds.

Theorem2 [1] (Lagrange's Mean Value Theorem) If the function $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable within the open interval (a,b), then there is at least one point ξ ($a<\xi<b$), such that the equation $f'(\xi)=\dfrac{f(b)-f(a)}{b-a}$ or $f'(\xi)(b-a)=f(b)-f(a)$ holds.

Theorem3 [1] (Cauchy's Mean Value Theorem) If the functions $f(x)$ and $F(x)$ satisfy being continuous on the closed interval $[a,b]$ and differentiable within the open interval (a,b), and $\forall x\in(a,b)$, $F'(x)\neq\mathbf{0}$,then there is at least one point ξ within the interval (a,b),such that $\dfrac{f'(\xi)}{F'(\xi)}=\dfrac{f(b)-f(a)}{b-a}$ holds.

II. Construction Method and Application of Auxiliary Function in Proof Problems

Although the conditions and conclusions of the three major mean value theorems are different, the connections between them are close. Therefore, it is particularly important to analyze the characteristics of the problem, including the structural features, conditions and conclusions of the problem. According to the characteristics of the problem, an appropriate auxiliary function can be constructed, and finally the corresponding mean value theorem can be used for proof. The construction of the auxiliary function mainly includes the original function method, the differential equation method, the constant k value method, the exponential factor method, etc [2]. These methods have their own characteristics. In practical applications, an appropriate method for constructing the auxiliary function can be selected according to the conditions, conclusions, and structure of the problem, which can effectively prove the differential mean value theorem.

Analyzing the conditions, conclusions, and structural characteristics of the problem often gives ideas and methods for constructing the auxiliary function.

(i) Integral Original Function Method

If the problem involves the situation of " $f(\xi)$ and $f(\xi)$ " two functions, the integral original function method is generally used to construct the auxiliary function.

The specific method is briefly described: Arrange the formula to be proved into the form $\varphi(\xi)=0$ (generally without fractions), let $F'(\xi)=\varphi(\xi)$, integrate both sides of the equation, $F(\xi)=\int\varphi(\xi)d\xi$, and then $F(x)$ is the constructed auxiliary function.

Example 1: Suppose $f(x), g(x)$ is continuous on the closed interval $[a,b]$ and differentiable within the open interval (a,b), and $g'(x)\neq 0$, prove: $\exists\xi,\xi\in(a,b)$, such that $\frac{f(\xi)-f(a)}{g(b)-g(\xi)}=\frac{f'(\xi)}{g'(\xi)}$.

Analysis: The two functions $f(\xi)$ and $g(\xi)$ appear in the problem, and the original function integral method can be considered to construct the auxiliary function.

The construction process of the auxiliary function is as follows:

(a) Arrange the conclusion as:

$$g'(\xi)f(\xi)+f'(\xi)g(\xi)-f(a)g'(\xi)-g(b)f'(\xi)=0;$$

(b) Let: $F'(\xi)=g'(\xi)f(\xi)+f'(\xi)g(\xi)-f(a)g'(\xi)-g(b)f'(\xi)$;

(c) Integrate both sides of the above equation to get：

$$F(\xi)=\int(g'(\xi)f(\xi)+f'(\xi)g(\xi)-f(a)g'(\xi)-g(b)f'(\xi))d\xi$$
$$=\int f(\xi)dg(\xi)+\int g(\xi)df(\xi)-f(a)g(\xi)-g(b)f(\xi)$$
$$=f(\xi)g(\xi)-\int g(\xi)df(\xi)+\int g(\xi)df(\xi)-f(a)g(\xi)-g(b)f(\xi)$$
$$=f(\xi)g(\xi)-f(a)g(\xi)-g(b)f(\xi);$$

(d) It can be known that the auxiliary function is:

$$F(x)=f(x)g(x)-f(a)g(x)-g(b)f(x);$$

Proof: Let $F(x)=f(x)g(x)-f(a)g(x)-g(b)f(x)$, then $F(x)$ is continuous on the closed interval $[a,b]$ and differentiable within the open interval (a,b), and $F(a)=F(b)=-f(a)g(b)$, so according to Rolle's Mean Value Theorem, there exists a point ξ within the interval (a,b) such that $F'(\xi)=0$, that is, $g'(\xi)f(\xi)+f'(\xi)g(\xi)-f(a)g'(\xi)-g(b)f'(\xi)=0$, that is $\frac{f(\xi)-f(a)}{g(b)-g(\xi)}=\frac{f'(\xi)}{g'(\xi)}$.

Example2 [3]: The proof of Lagrange's Mean Value Theorem.

Analysis: The textbook proves it by introducing the auxiliary function $\varphi(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$,The process is as follows:

Let $\varphi(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$, $\varphi(x)$ is continuous on the closed interval $[a,b]$ and differentiable within the open interval (a,b), and $\varphi(a)=\varphi(b)=0$, and $\varphi'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$,According to Rolle's Mean Value Theorem, it can be known that there is at least one point ξ within the interval (a,b), such that $f'(\xi)-\frac{f(b)-f(a)}{b-a}=0$, that is $\frac{f(b)-f(a)}{b-a}=f'(\xi)$,and thus $f(b)-f(a)=f'(\xi)(b-a)$.

The following is a method of constructing the auxiliary function by the original function integral method to prove Lagrange's Mean Value Theorem.

The process of constructing the auxiliary function is as follows: Arrange the formula $\frac{f(b)-f(a)}{b-a}=f'(\xi)$ as $f'(\xi)-\frac{f(b)-f(a)}{b-a}=0$,both sides at the same time, and get $f(\xi)-\frac{f(b)-f(a)}{b-a}\xi=0$, so $F(x)=f(x)-\frac{f(b)-f(a)}{b-a}x$ is the constructed auxiliary function.

The proof process is as follows:

Let $F(x)=f(x)-\frac{f(b)-f(a)}{b-a}x$, $F(x)$ is continuous on the closed interval $[a,b]$ and differentiable within the open interval (a,b), and $F(a)=F(b)=\frac{bf(a)-af(b)}{b-a}$, According to Rolle's Mean Value Theorem, it can be known that there is at least one point ξ within the interval (a,b) such that $F'(\xi)=0$, and thus $f'(\xi)-\frac{f(b)-f(a)}{b-a}=0$,that is $\frac{f(b)-f(a)}{b-a}=f'(\xi)$,that is $f(b)-f(a)=f'(\xi)(b-a)$.

(ii) Differential Equation Method

(a) If the equality " $f'(\xi)-kf(\xi)$ " appears in the problem, the differential equation method can be used to construct the auxiliary function.

The specific method is briefly described: Regard the expression to be proved as a differential equation $\varphi(f'(\xi),f(\xi),\xi)=0$, solve it from $F(y,x)=0$, then ignore the constant term, and replace it with $F(f(x),x)=0$, which is the constructed auxiliary function.

Example 3: $f(x)$ is known that is continuous $[a,b]$ and $f(a)=f(b)=0$, prove that $\exists\xi,\xi\in(a,b)$ such that $\frac{f'(\xi)}{-2\xi}=f(\xi)$.

Analysis: The problem appears $f'(\xi)-kf(\xi)$ and the differential equation method can be considered to construct the auxiliary function. The construction process is as follows:

a) Arrange the fraction as $f'(\xi)+2\xi f(\xi)=0$;

b) Separate the variables to get: $\frac{1}{y}dy=-2\xi d\xi$;

c) Integrate both sides at the same time to get $\int\frac{1}{y}dy=\int-2\xi d\xi$, that is $\ln y=-\xi^2+C$;

d) If $C=0$, there is $\ln y=-\xi^2$, that is $y=e^{-\xi^2}$, and further get $ye^{\xi^2}-1=0$, ignore the constant term, then $F(x)=f(x)e^{x^2}$ is the auxiliary function.

Proof: Let $F(x)=f(x)e^{x^2}$, $F(x)$ is continuous on the closed interval $[a,b]$ and differentiable within the open interval (a,b), and $F(a)=F(b)=0$, so according to Rolle's Mean Value Theorem, there exists a point ξ within the interval (a,b),such that $F'(\xi)=0$, that is $F'(\xi)=f'(\xi)e^{\xi^2}+2\xi e^{\xi 2}f(\xi)=0$,that is $f'(\xi)+2\xi f(\xi)=0$, $\frac{f'(\xi)}{-2\xi}=f(\xi)$, so it is proved.

(b) If the form of " $F(\xi,f(\xi),f'(\xi))=0$ " appears in the problem, the differential method can be used to construct the auxiliary function.

The specific method is briefly described: On The left side of equality $F(\xi,f(\xi),f'(\xi))=0$ is regarded as the derivative of a certain function $g(x)$ at the point ξ, that is $g'(\xi)=0$, then the general solution of the original differential equation is $g(x)=C$, and thus, the constant term is the auxiliary function to be found, that is $F(x)=g(x)$.

Example 4: Suppose the function $f(x)$ is continuous on the closed interval $[0,\pi]$ and differentiable on the open interval $(0,\pi)$,Prove: There exists at least one point ξ within the open interval $(0,\pi)$ such that. $f'(\xi)\sin\xi=-f(\xi)\cos\xi$.

Analysis: The differential method can be used to construct the auxiliary function.

The construction process is as follows:

a) Separate $f'(\xi)\sin\xi=-f(\xi)\cos\xi$ variables to get: $\frac{dy}{y}=-\frac{\cos\xi}{\sin\xi}d\xi$;

b) Integrate both sides at the same time to have, $\ln y=-\ln\sin\xi+C$, that is $\ln y\sin x=C$;

c) The general solution $y\sin x=C$, that is the auxiliary function. $F(x)=f(x)\sin x$;

Proof: Let $F(x)=f(x)\sin x$, $F(x)$ is continuous on the closed interval $[\mathbf{0},\boldsymbol{\pi}]$ and differentiable within the open interval $(\mathbf{0},\boldsymbol{\pi})$, and $F(\mathbf{0})=F(\boldsymbol{\pi})$, According to Rolle's Mean Value Theorem, it is known that within the interval $(\mathbf{0},\boldsymbol{\pi})$,there is at least one point ξ such that $F'(\xi)=\mathbf{0}$, that is $F'(\xi)=f'(\xi)\sin\xi+f(\xi)\cos\xi=\mathbf{0}$,that is $f'(\xi)\sin\xi=-f(\xi)\cos\xi$.

(iii) Method of Moving Terms

If the form of " $F(\xi, f(\xi), f'(\xi))=C$ " appears in the problem, the method of moving terms can be used to construct the auxiliary function.

The specific method is briefly described: For the proof of the constant equation of the form $F(\xi, f(\xi), f'(\xi))=C$, the corollary of Lagrange's Mean Value Theorem is the best choice. Therefore, only need to move the constant term and the function term to the left and right ends of the equation respectively, and the obtained function term is the constructed auxiliary function.

Example5 [4]: Prove $\mathbf{2}\arctan x+\arcsin\frac{\mathbf{2}x}{\mathbf{1}+x^2}-\boldsymbol{\pi}=\mathbf{0}(x\geq\mathbf{1})$.

Analysis: Arrange the formula $\mathbf{2}\arctan x+\arcsin\frac{\mathbf{2}x}{\mathbf{1}+x^2}=\boldsymbol{\pi}$ into an equation, then the auxiliary function is: $F(x)=\mathbf{2}\arctan x+\arcsin\frac{\mathbf{2}x}{\mathbf{1}+x^2}$.

Proof: a) When $x=\mathbf{1}$, the left end $\mathbf{2}\arctan\mathbf{1}+\arcsin\mathbf{1}=\mathbf{2}\cdot\frac{\boldsymbol{\pi}}{\mathbf{4}}+\frac{\boldsymbol{\pi}}{\mathbf{2}}=\boldsymbol{\pi}$, $\boldsymbol{\pi}-\boldsymbol{\pi}=\mathbf{0}$,The proof is completed.

b) When $x>\mathbf{1}$,let the auxiliary function is $F(x)=\mathbf{2}\arctan x+\arcsin\frac{\mathbf{2}x}{\mathbf{1}+x^2}$,

then $F'(x)=\frac{\mathbf{2}}{\mathbf{1}+x^2}+\frac{\mathbf{1}}{\sqrt{\mathbf{1}-(\frac{\mathbf{2}x}{\mathbf{1}+x^2})^2}}\frac{\mathbf{2}(\mathbf{1}+x^2)-\mathbf{2}x\cdot\mathbf{2}x}{(\mathbf{1}+x^2)^2}=\mathbf{0}$, so $F(c)=\mathbf{0}$, and because $F(\sqrt{\mathbf{3}})=\mathbf{2}\arctan\sqrt{\mathbf{3}}+\arcsin\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}=\boldsymbol{\pi}$, therefore $c=\boldsymbol{\pi}$,and thus $\mathbf{2}\arctan x+\arcsin\frac{\mathbf{2}x}{\mathbf{1}+x^2}-\boldsymbol{\pi}=\mathbf{0}$, the proof is completed.

III. Conclusion

The methods of constructing the auxiliary function in the proof problems of the differential mean value theorem are diverse and flexible. In the proof process of specific problems, specific problems should be analyzed specifically. Several methods of constructing the auxiliary function are discussed and summarized, which have a certain role in improving students' mathematical thinking ability.

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