MATHEMATICAL SCIENCES

PROBLEM OF BENDING OF A TRANSVERSELY ISOTROPIC PLATE

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https://doi.org/10.5281/zenodo.14577666

Abstract
The thermoelastic equilibrium of a transtropic plate is considered when the symmetry conditions are satisfied on the lateral faces, and special conditions are specified on the upper and lower faces. The exact solution to the problem was found using general solutions and the method of separation of variables (Fourier method).

Keywords: thick plate, thermoelasticity, elastic equilibrium, transtropy, exact solutions, method of separation of variables.

Introduction. The theory of thick plates, based on the equations of equilibrium and continuity of an elastic isotropic body, on which only surface forces act, was developed by Mitchell and discussed in detail by Love [1]. In subsequent studies, the solutions of the authors mentioned were simplified and generalized, and the necessary technical theories were developed for calculating the slabs. In this direction, the technical theories of Reisner-Bohl, B.T. Vlasov, V.Z. Vlasov, and L. Donel should be noted [2]. They work under certain conditions, but calculations of thick plates are essentially close to the corresponding three-dimensional problems. For their part, three-dimensional problems for all technical theories play the role of the highest arbiter. And therefore great importance is given to the construction of exact (analytical) solutions of three-dimensional problems, which in a certain sense are analogous to problems of thick plates.

For inhomogeneous transversely isotropic bodies bounded by coordinate surfaces of generalized cylindrical coordinates, in works [3,4], exact (analytical) solutions were obtained for boundary-contact problems of thermoelasticity.

On the lateral faces of the body, conditions of symmetric or antisymmetric continuation are specified, and on the upper and lower faces, special conditions are specified. Solutions are obtained for the case when the curvilinear coordinate parallelepiped is multilayered.

The method we use to construct the exact solution is based on a record modified by the equilibrium equations and boundary-contact conditions, as well as on the convenient use of the elastic field ingredients by decomposing them into the divergence and rotor of the displacement vector. All components of the elastic field are expressed through u, v, w, B, K where B and K are, respectively, the analogy of the divergence and one of the components of the rotor of the displacement vector.

2. main part

In a Cartesian rectangular coordinate system, we consider a thick plate that occupies an area $\Pi = \{(x, y, z) \in R : 0 \le x \le x_1, 0 \le y \le y_1, 0 \le z \le z_1,\}$.

The plate has the shape of a rectangular parallelepiped. Transversely isotropic (Transtropic) homogeneous plate with isotropy plane $z = const.$ The plate is affected by a stationary temperature field, i.e. the temperature field does not depend on time; — there are no mass forces, then the equilibrium equations in the Cartesian rectangular coordinate system have the following form [1]

$$\begin{aligned} &a)\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \\ &\text{b)}\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0, \qquad (1) \\ &c)\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0. \end{aligned}$$

$\sigma_x, \sigma_y, \sigma_z$ -normal stresses, $\tau_{xy} = \tau_{yx}, \tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}$ -tangential stresses.

Generalized Hooke's law, also known as the physical principle for a transversally isotropic medium, is stated as follows [5]

$$\sigma_x = c_1 e_{xx} + \left(c_1 - 2c_5\right) e_{yy} + c_3 e_{zz} - k_{10} T = c_1 \left(\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} \right) - 2c_5 \frac{\partial \mathrm{v}}{\partial y} + c_3 \frac{\partial \mathrm{w}}{\partial z} - k_{10} T,$$

$$\sigma_y = c_1 e_{yy} + \left(c_1 - 2c_5\right) e_{xx} + c_3 e_{zz} - k_{10} T = c_1 \left(\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} \right) - 2c_5 \frac{\partial u}{\partial x} + c_3 \frac{\partial \mathrm{w}}{\partial z} - k_{10} T,$$

$$\sigma_z = c_2 e_{zz} + c_3 \left(e_{yy} + e_{xx} \right) - k_{20} T = c_1 \frac{\partial \mathrm{w}}{\partial z} + c_3 \left(\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} \right) - k_{20} T.$$

$$\tau_{xz} = \tau_{zx} = c_4 e_{xz} = c_4 \left(\frac{\partial \mathrm{w}}{\partial x} + \frac{\partial u}{\partial z} \right), \tag{2}$$

$$\tau_{yz} = \tau_{zy} = c_4 e_{yz} = c_4 \left(\frac{\partial \mathrm{w}}{\partial y} + \frac{\partial \mathrm{v}}{\partial z} \right),$$

$$\tau_{xy} = \tau_{yx} = c_5 e_{xy} = c_5 \left(\frac{\partial \mathrm{v}}{\partial x} + \frac{\partial u}{\partial y} \right).$$

Here $e_{xx}, e_{yy}, e_{zz}, e_{xz} = e_{zx}, e_{yx} = e_{xy}, e_{xy} = e_{yx}$ - deformations; $u, \mathrm{v}, \mathrm{w}$ -components of the displacement vector $\vec{U}$; $c_i \left(i = \overline{1,5} \right)$ elastic characteristics (their expressions through technical characteristics $E_1, E_2, \nu_1, \nu_2, \mu$ see [5]); k_1, k_2 - coefficients of linear thermal expansion in the plane of isotropy and along z ; T is the temperature in an elastic body obeying the equation

$$\Delta_2 T + \frac{\lambda_1}{\lambda_2} \frac{\partial^2 T}{\partial z^2} = 0 \tag{3}$$

and the corresponding boundary conditions. λ_1, λ_2 - linear thermal conductivity coefficients in the isotropy plane and along z . $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ -.

Taking into account (1) and (2), the equilibrium equations can be written as follows[6]:

$$a)\ \frac{c_3}{c_1} \frac{\partial K}{\partial z} + \frac{c_1 c_2 - c_3^2}{c_1} \frac{\partial^2 \mathrm{w}}{\partial z^2} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial \alpha} = \frac{c_3 k_{20} - c_3 k_{10}}{c_1} \frac{\partial T}{\partial z},$$

$$b)\ \frac{\partial K}{\partial x} - \frac{\partial B}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0,$$

$$c)\ \frac{\partial B}{\partial x} + \frac{\partial K}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0, \tag{4}$$

$$d)\ \frac{\partial \tau_{zy}}{\partial x} - \frac{\partial \tau_{zx}}{\partial y} - \frac{c_4}{c_5} \frac{\partial B}{\partial z} = 0,$$

where

$$a)\ K = c_1 \left(\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} \right) + c_3 \frac{\partial \mathrm{w}}{\partial z} - k_{10} T,$$

$$b)\ B = c_5 \left(\frac{\partial \mathrm{v}}{\partial x} - \frac{\partial u}{\partial y} \right),$$

$$c)\ \tau_{zx} = c_4 \left(\frac{\partial u}{\partial z} + \frac{\partial \mathrm{w}}{\partial x} \right), \tag{5}$$

$$d)\ \tau_{zy} = c_4 \left(\frac{\partial \mathrm{v}}{\partial z} + \frac{\partial \mathrm{w}}{\partial y} \right).$$

The boundary conditions that further define the class of boundary value problems under consideration have the following form:

When $x = x_j : \begin{cases} a) \quad \dfrac{\partial T}{\partial x} = 0, \quad u = 0, \quad \tau_{zx} = 0, \quad B = 0, \\ b) \quad T = 0, \quad K = 0, \quad \mathrm{v} = 0, \quad \mathrm{w} = 0. \end{cases}$ (6)

When: $y = y_j : \begin{cases} a) \quad \dfrac{\partial T}{\partial y} = 0, \quad \mathrm{v} = 0, \quad \tau_{zy} = 0, \quad B = 0, \\ b) \quad T = 0, \quad K = 0, \quad \mathrm{u} = 0, \quad \mathrm{w} = 0. \end{cases}$ (7)

When: $z = z_j : \begin{cases} a) \quad \dfrac{\partial T}{\partial y} = F_{j1}(x,\mathrm{y}), \quad b) \ T = F_{j2}(x,\mathrm{y}), \\ c) \quad \dfrac{\partial T}{\partial \alpha} + \Theta_j T = F_{j3}(x,\mathrm{y}). \end{cases}$ (8)

When: $z = z_j : \begin{cases} a) \quad \sigma_Z = F_{j4}(x,\mathrm{y}), \quad \tau_{zx} = F_{j5}(x,\mathrm{y}), \quad \tau_{zy} = F_{j6}(x,\mathrm{y}), \\ \mathrm{b}) \quad \mathrm{w} = f_{\mathrm{j}1}(x,\mathrm{y}), \quad u = f_{j2}(x,\mathrm{y}), \quad \mathrm{v} = f_{\mathrm{j}3}(x,\mathrm{y}), \\ c) \quad \sigma_z = F_{j4}(x,\mathrm{y}), \quad u = f_{j2}(x,\mathrm{y}), \quad \mathrm{v} = f_{\mathrm{j}3}(x,\mathrm{y}), \\ d) \quad \mathrm{w} = f_{j1}(x,\mathrm{y}), \quad \tau_{zx} = F_{j5}(x,\mathrm{y}), \quad \tau_{zy} = F_{\mathrm{j}6}(x,\mathrm{y}). \end{cases}$ (9)

The following conditions are imposed on the $F_{ip}(x,y)$ $\left(p=\overline{1,6}\right)$, $f_{is}(x,y)\left(s=\overline{1,3}\right)$

functions $F_{ip}(x,y) \in C^{1,q}(z_i)$, $f_{is}(x,y) \in C^{2,q}(z_i)$. $C^{i,q}(z_i)$ -class of i-times continuously differentiable functions on the surface of the plate, where the -th derivative belongs to the Hölder class $C^{0,q}(z_i), \dfrac{1}{2} < q < 1$. These conditions allow us to represent the functions by absolutely and uniformly convergent Fourier series [7].

The purpose of this work is to construct a hegudary solution of boundary value problems (3), (4) ,(5), (6), (7) ,(8), (9) the refore we define the concept of regularity.

The solution of the system (4),(5) defined by the functions $u, \mathrm{v}, \mathrm{w}$ will be called regular if the functions $u, \mathrm{v}, \mathrm{w}$ are three times continuously differentiable in the region $\overline{\Pi}$, where $\overline{\Pi}$ is the region Π together with the boundaries $x=0, x=x_1, y=0, \mathrm{y}=y_1$, and on the surface $z=z_0$ 0=0 and $z=z_1$ can be represented with their first and second derivatives by absolutely and uniformly convergent Fourier series in the eigenfunctions of the corresponding Sturm-Liouville problem.

For the class of boundary value problems of thermoelasticity under consideration, the general solution in the class of regular functions is represented in the form [6]:

$$\begin{aligned} \mathrm{u} &= \frac{\partial}{\partial x}\left(\psi_2 + \frac{1}{2c_4}\psi_1\right) + \frac{1}{c_5}\frac{\partial \psi_0}{\partial y}, \\ \mathrm{v} &= \frac{\partial}{\partial y}\left(\psi_2 + \frac{1}{2c_4}\psi_1\right) - \frac{1}{c_5}\frac{\partial \psi_0}{\partial x}, \\ \mathrm{w} &= -\frac{\partial}{\partial z}\left(\psi_2 + \frac{1}{2c_4}\psi_1\right) + \frac{1}{c_4}\frac{\partial \psi_1}{\partial z}. \end{aligned} \tag{10}$$

Here

$$\begin{aligned} &a)\Delta_2\psi_0 + \frac{c_4}{c_5}\frac{\partial^2\psi_0}{\partial z^2} = 0, \\ &b)\Delta_2\psi_1 + \gamma_1\frac{\partial^2\psi_1}{\partial z^2} - \gamma_2\frac{\partial^2\psi_2}{\partial z^2} - \gamma_3 T = 0, \\ &c)\Delta_2\psi_1 + \gamma_4\frac{\partial^2\psi_1}{\partial z^2} + \gamma_5\Delta_2\psi_1 - \gamma_6\frac{\partial^2\psi_2}{\partial z^2} - \gamma_7 T = 0, \end{aligned} \tag{11}$$

Where

$$\gamma_1 = \frac{c_1c_2 - c_3^2 - 2c_3c_4}{2c_1c_4}, \gamma_2 = \frac{c_1c_2 - c_3^2}{c_1}, \gamma_3 = \frac{c_1k_{20} - c_3k_{10}}{c_1}, \gamma_4 = \frac{c_3 + 2c_4}{c_1}, \gamma_5 = 2c_4,$$
$$\gamma_6 = \frac{2c_3c_4}{c_1}, \gamma_7 = \frac{2c_4k_{10}}{c_1}. \qquad (12)$$

As an illustration, in the domain $\Pi = \{(x, y, z) \in R : 0 \le x \le x_1, 0 \le y \le y_1, 0 \le z \le z_1,\}$, we consider the following boundary value problem.

The plate has the shape of a rectangular parallelepiped. On the lateral faces, the conditions of antisymmetric continuation are set, i.e. on the lateral faces we have the following boundary conditions

$$\text{When } x = x_j: \ T = 0, \mathrm{u} = 0, \ \tau_{zx} = 0, B = 0; \qquad (13)$$

$$\text{When: } y = y_j: \quad T = 0, \mathrm{v} = 0, \ \tau_{zy} = 0, B = 0; \qquad (14)$$

$$\text{When: } z = z_j: \ \sigma_Z = F_{j4}(x, \mathrm{y}), \ \tau_{zx} = F_{j5}(x, \mathrm{y}), \ \tau_{zy} = F_{j6}(x, \mathrm{y}). \qquad (15)$$

At first glance, it seems as if the problem statement is artificial, but it is not. As Donel writes in his work, the boundary conditions () are similar to the case when the edge of a thick plate rests on short rollers []. Since its friction is quite large, it prevents the roller from moving along the axis. Regarding the conditions at the ends, we note that the external load can be represented as a superposition of symmetric and antisymmetric loads.

Let us consider an elastic field when the $B = c_5\left(\frac{\partial \mathrm{v}}{\partial x} - \frac{\partial u}{\partial y}\right)$, component of the displacement rotor in the entire region is identically equal to zero. We will call such a field the field of the generalized parallel rotor of the displacement vector.

The equilibrium equations will take the following form

$$a)\ \frac{c_3}{c_1}\frac{\partial K}{\partial z} + \frac{c_1c_2 - c_3^2}{c_1}\frac{\partial^2 \mathrm{w}}{\partial z^2} + \frac{\partial \tau_{\mathrm{zx}}}{\partial x} + \frac{\partial \tau_{\mathrm{zy}}}{\partial \alpha} = \frac{c_3k_{20} - c_3k_{10}}{c_1}\frac{\partial T}{\partial z},$$
$$b)\ \frac{\partial K}{\partial x} + \frac{\partial \tau_{\mathrm{zx}}}{\partial z} = 0,$$
$$c)\ \frac{\partial K}{\partial y} + \frac{\partial \tau_{\mathrm{zy}}}{\partial z} = 0,$$
$$d)\ \frac{\partial \tau_{\mathrm{zy}}}{\partial x} - \frac{\partial \tau_{\mathrm{zx}}}{\partial y} = 0. \qquad (16)$$

Assuming that $T = \frac{\partial^2 \tilde{T}}{\partial z^2}$, $\psi_1 = \Psi_1 + \delta\psi_2, \psi_2 = \psi_2$, after obvious, simple transformations (meaning elementary operations on equations, in particular their linear combinations) we obtain the following system

$$\Delta_2\Psi_1 + \bar{\gamma}_1\frac{\partial^2\Psi_1}{\partial z^2} = \bar{\gamma}_3\frac{\partial^2\tilde{T}}{\partial z^2},$$
$$\Delta_2\psi_2 + \bar{\gamma}_4\frac{\partial^2\psi_2}{\partial z^2} = \bar{\gamma}_6\frac{\partial^2\Psi_1}{\partial z^2} - \bar{\gamma}_7\frac{\partial^2\tilde{T}}{\partial z^2},$$

Where

$$\delta = \sqrt{\frac{\gamma_1}{4\gamma_4}}.$$

In the case under consideration $\psi_0 \equiv 0$, and the functions Ψ_1 and ψ_2 have the following form

$$\Psi_1(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_{1mn}e^{-p_1z}sh(p_1z)\cdot\sin\left(\frac{m\pi}{x_1}x\right)\sin\left(\frac{m\pi}{y_1}y\right), \qquad (17)$$
$$\psi_2(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\psi_{2mn}e^{-p_2z}sh(p_2z)\cdot\sin\left(\frac{m\pi}{x_1}x\right)\sin\left(\frac{m\pi}{y_1}y\right),$$

Where $p_1 = \gamma_3^{-0.5}\cdot p(m,n) \ge 0, p_2 = \gamma_4^{-0.5}\cdot p(m,n) \ge 0, p^2 = m^2 + n^2, m = \frac{\pi\bar{m}}{x_1}, n = \frac{\pi\bar{n}}{y_1}$,

$\bar{m} = 1, 2, 3, \cdots, \bar{n} = 1, 2, 3, \cdots.$

for the function $\tilde{T}$ we will have

$$\tilde{T}(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{T}_{3mn} e^{-p_3 z} sh(p_3 z) \cdot \sin\left(\frac{m\pi}{x_1} x\right) \sin\left(\frac{m\pi}{y_1} y\right), \; p_3 = \lambda^{-0.5} p_{mn}.$$

The components of the elastic field expressed through these functions have the following form

$$\begin{aligned} K &= -\gamma_3 \frac{\partial^2 \psi_2}{\partial z^2} - \frac{\gamma_3}{2\gamma_1} z \frac{\partial^2 \psi_1}{\partial z^2} - \frac{\gamma_3}{\gamma_1} z \frac{\partial^2 \psi_1}{\partial z^2}, \\ \mathrm{u} &= 2\gamma_1 \gamma_4 \frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_1}{\partial x} + \gamma_4 z \frac{\partial^2 \psi_1}{\partial x \partial z}, \\ \mathrm{v} &= 2\gamma_1 \gamma_4 \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_1}{\partial y} + \gamma_4 z \frac{\partial^2 \psi_1}{\partial y \partial z}, \\ \mathrm{w} &= 2\gamma_1 \gamma_4 \frac{\partial \psi_2}{\partial z} + (\gamma_4 - 1) \frac{\partial \psi_1}{\partial z} + \gamma_4 z \frac{\partial^2 \psi_1}{\partial z^2}, \end{aligned} \tag{18}$$

Using (17), (18) and (2), we can obtain expressions for $\sigma_z, \tau_{zx}, \tau_{zy}$ when $z=z_i$ and we equate to the corresponding function given on these same faces, represented by trigonometric Fourier series. As a result, to determine the coefficients Ψ_{1mn}, Ψ_{2mn} we obtain an algebraic system, the determinant of which, when $p \to \infty, \Delta_i \neq 0$.

conclusion

The problem under consideration is nothing more than the bending of a hinged thick plate caused by a tangential load.

The general solution obtained enables us to confirm this, u and v are odd functions relative to z, that is, the middle surface, in our case on the surface z=0 the displacements u and v are equal to zero. But this is so only in the case when antisymmetric loads act on the ends of the plate.

The following works will consider the case when antisymmetric normal loads act on the ends, as well as the boundary-contact problem for a multilayer plate.

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