MATHEMATICAL SCIENCES

OSCILLATION CONDITIONS OF ONE TYPE OF ELLIPTIC EQUATIONS

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https://doi.org/10.5281/zenodo.15298387

Abstract

In this paper, the oscillatory properties of an elliptic equation are considered, that is, its ability to oscillate. As you know, an oscillating solution to an equation is one that changes its sign with an unlimited increase in the value of a variable.

One of the key tasks in studying these equations is to determine their oscillatory characteristics. The paper presents and proves theorems on the oscillation of solutions, which is of considerable interest to mathematicians.

Keywords: oscillatory properties, elliptic equation, function, inequality, theorem, proof.

The paper examines the oscillatory characteristics of the elliptic equation.

$$\Delta^{m+1}U + \Delta^m U + P(x)|U|^\lambda \operatorname{sign} U = 0 \tag{1}$$

Assuming that… ,$m \geq 1, \;\; \lambda > 1, \;\; x \in R_a^n$

$$R_a^n = \{x \in R^n = \{x: x \in R^n, |x| > a\}$$

And the theorem on the oscillation of all solutions to equation (1) has been proved if$n > 2m$,n is an odd number. Equation (1) was previously studied only at , $\lambda > 1$ [1] and for any *n* in [2].

Consider equation (1), where $\mathrm{R}_\mathrm{a}^\mathrm{n} = \{\mathrm{x} \in \mathrm{R}^\mathrm{n}: |\mathrm{x}| > \mathrm{a}\}$, .$a > 0$

The equation

$$\Delta^m U + P(x)|U|^\lambda \operatorname{sign} U = 0$$

was studied in more detail in [3, 4].

In [3], a necessary and sufficient condition for the oscillation of all solutions for $\lambda > 1$is given, as well as a theorem on the oscillation of all solutions for the value λ: $0 < \lambda \leq 1$

Sufficient oscillation conditions for $\lambda > 1$are also given in [4].

Definition 1. Function $U(x)$ is called the solution of equation (1) if $U(x) \in C^{2m}(R_a^n)$ and satisfies equation (1).

Definition 2. The solution of equation (1) is called oscillating if it changes sign $\forall a > 0$ в R_a^n, and non-oscillating otherwise.

Let $U(x)$be the solution of equation (1). Denote

$$U(x) = \frac{1}{\omega_n} \int\limits_{|\varphi|=1} U(r,\varphi)\, d\varphi,$$

(r,φ) 254 – spherical coordinates of point $x \in R^n$, ω_n – area $(n-1)$ the dimensional unit sphere$|\varphi| = 1, \;\; \varphi = (\varphi_1, \varphi_2, \dots, \varphi_{n-1})$.

If $U(x) \geq 0$, then according to Jensen's inequality [5, p.32] for $U(x) \geq 0$ и $\lambda > 1$and we have

therefore, it is valid for $U(x) \geq 0$

So in this case

$$L\omega + P(r)U^\lambda \leq 0 \tag{2}$$

L - an ordinary linear differential operator of order and

$$\omega(r) = \Delta_r U + U, \qquad \Delta_r = \frac{d^2}{dr^2} + \frac{n_1}{r}\frac{d}{dr}, \tag{3}$$

Δ_r – the radial part of the Laplace operator.

Lemma 1. If $U(x) \geq 0$ is R_a^n, then $\omega(r) \geq 0$ и $\left(r^{\frac{n-2}{2}}\omega(r)\right)' \geq 0$ в R_a^n, $a > 0$.

Proof. First, we prove that $\omega(r) \geq 0$ for $r \geq a$. In equation (3), we replace $U = F(r)\, r^{\frac{2-n}{2}}$, then we get

$$F'' + \frac{1}{r}F' + \left(1 - \frac{(n-2)^2}{4r^2}\right)F = \omega(r)\, r^{\frac{n-2}{2}} = \omega_1(r). \tag{4}$$

Let $\omega(r) < 0$ be at $r > r_1 > a$. Then from (4) we have

$$F'' + \frac{1}{r}F' + \left(1 - \frac{(n-2)^2}{4r^2}\right)F \leq 0$$

Or

$$(rF')' + r\left(1 - \frac{(n-2)^2}{4r^2}\right)F \leq 0.$$

Hence, at a high r, we get

$$(rF')' \leq 0.$$

In this case, there are two possible options: $rF' \leq 0$, or $rF' \geq 0$. If $F' \leq 0$, then the function is negative and does not increase, therefore

$$rF' < -c, \qquad c = const > 0 \quad \text{at} \quad r > r_2 > r_1.$$

From here we find

$$F(r) < -c\ln r + c_1 \to -\infty \quad \text{at} \quad r \to \infty,$$

which contradicts the condition $F(r) > 0$. So, $rF' \geq 0$. Now let's prove that $F''(r) \geq 0$. Let $F''(r) < 0$. By virtue of the conditions $F' \geq 0$, $F \geq 0$ from (4), we have

$$F'' < -\frac{1}{2}F.$$

From here we easily come to the inequality

$$F'(r) < -\frac{1}{2}F(r_1)(r - r_1) + F'(r_1) \to -\infty$$

for $r \to \infty$, and this contradicts the already proven fact that $F' > 0$. Thus, it was proved that $\omega(r) > 0$

Now let's prove that $\omega_1' = \left(r^{\frac{n-2}{2}}\omega\right)' \geq 0$. If $U(r) \geq 0$, then from (2) we get that $L\omega \leq 0$. In this inequality, we replace $t = \ln r$, $\omega = z\exp\left(\frac{2m-n}{2} - t\right)t$. Then the roots of the characteristic polynomial of the equation $L\omega = 0$ are [6, 7]

$$\pm\left(\frac{n - 2m + 4i}{2}\right), \qquad i = 0, 1, 2, \dots, m-1.$$

It follows that

$$Lz = \sum_{i=1}^{m} \beta_{2i}\, z^{(2i)}, \qquad \beta_{2i+1} = 0, \qquad \beta_{2i} = (-1)^i|\beta_{2i}|, \qquad i = 0, 1, 2, \dots, m.$$

Since $z \geq 0$, we have

$$L_1 z = \sum_{i=1}^{m} \beta_{2i}\, z^{(2i)} \leq 0.$$

For the characteristic polynomial of the equation $L_1 z = 0$ all roots are valid. Moreover, as follows from the results of [4, tasks 36, 47], there are] $m-1$ different roots. Since the polynomial in question is an even function, if α is its root, then $-\alpha$ will be its root. Thus, the characteristic polynomial has different roots and one double root equal to 0.

Let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{m-1}$
be the roots of this characteristic polynomial, then

$$-\alpha_{m-1} < -\alpha_{m-2} < \cdots < -\alpha_1 < \alpha_1 < \alpha_2 < \cdots < \alpha_{m-1}.$$

Thus, the fundamental system of solutions of the equation $L_1 z = 0$ has the form

$$z_1 = 1, \quad z_2 = t, \quad z_3 = \exp(-\alpha_{m-1} t), \quad z_4 = \exp(-\alpha_{m-2} t), \quad \dots, \quad z_{zm} = \exp(\alpha_{m-1} t).$$

Now we note the following fact from the theory of ordinary differential equations. If the linear equation is $L_2(y) = 0$ is non-oscillatory, then it can be represented as [8, 9]:

$$L_2(y) = q_{2m}(t)\frac{d}{dt}q_{2m-1}(t)\frac{d}{dt}\dots q_2(t)\frac{d}{dt}q_1(t)\frac{d}{dt}q_0 y, \tag{5}$$

and

$$q_0(t) = \frac{1}{W_1(t)}, \quad q_k(t) = \frac{W_k^2(t)}{W_{k-1}(t)W_{k+1}(t)}, \quad k = 1, 2, \dots, 2m-1; \quad q_{2m}(t) = \frac{W_{2m}(t)}{W_{2m-1}(t)} \tag{6}$$

here $W_k(t)$ is the Vronsky determinant of solutions $y_1, y_2, \dots, y_k$, with $W_k(t) > 0$ $(k = 1, 2, \dots, 2m)$, $W_0(t) = 1$. For our case, the functions $q_k(t)$ are easily found

$$q_0(t) = 1, \quad q_1(t) = 1, \quad q_2(t) = \exp(-\alpha_{m-1} t),$$
$$q_k(t) = \exp(\alpha_{m-k+1} - \alpha_{m-k+2}), \quad (k = 3, 4, \dots, m-1),$$
$$q_{m+1}(t) = \exp(-2\alpha_1 t), \quad q_{2m}(t) = \exp(\alpha_{m-1} t).$$
$$q_{m+k}(t) = \exp(\alpha_{k-1} - \alpha_k)\, t, \quad (k = 2, 3, \dots, m-1)$$

Let's introduce the notation

$$V_0 = q_0 \cdot z, \quad V_1 = q_1(t)\frac{d}{dt}q_0 z, \quad V_k = q_k(t)\frac{d}{dt}V_{k-1},$$
$$(k = 1, 2, \dots, 2m-1), \quad V_{2m} = \frac{d}{dt}V_{2m-1}. \tag{7}$$

Offer. If $z \geq 0$, $V_{2m} \leq 0$, then there are numbers $tul = 2k+1$ such that inequalities occur

$$V_k(t) \geq 0, \qquad k = 1, 2, \dots, l, \qquad l = 2k+1, \qquad 0 \leq k \leq m-1$$
$$(-1)^i V_{i+l} \geq 0, \quad i = l+1, l+2, \dots, 2m-l \tag{8}$$

Proof. From $z \geq 0$ and $V_{2m} \leq 0$ follows $V_{2m-1}(t) \geq 0$, or $V_{2m-1}(t) \leq 0$. If $V_{2m-1}(t) \leq 0$, then the function $V_{2m-1}(t$ is non-positive and decreasing, so $V_{2m-1}(r) < -c$, $c = const > 0$ for $t \geq t_1$, integrating the inequality

$$\frac{dV_{2m-r}}{dt} < -\frac{c}{q_{2m-1}(t)}$$

from t_1 to t, we get

$$V_{2m-2}(t) < V_{2m-2}(t) - c\int_{t_1}^{t}\frac{d\tau}{q_{2m-1}(\tau)} \to -\infty$$

for $t \to \infty$, therefore $V_{2m-2}(t) < 0$ for $t \geq t_2 > t_1$. Similarly, we obtain that $V_{2m-3} < 0$ for $t \geq t_3 > t_2$, etc. Finally, we obtain that $V_0(t) < 0$ for $t \geq t_{2m}$, which contradicts the condition of lemma $z(t) > 0$.

Thus, $V_{2m-1}(t) \geq 0$. Now either $V_{2m-2} \geq 0$ or $V_{2m-2}(t) \leq 0$.. If $V_{2m-2}(t) \geq 0$, then it is easy to obtain that $V_i(t) \geq 0$, i.e. $l = 2m-1$. If $V_{2m-2}(t) \leq 0$, then $V_{2m-3}(t) \geq 0$. Then either $V_{2m-4}(t) \leq 0$, or $V_{2m-4}(t) \geq 0$, etc.

If $V_{2m-4}(t) \geq 0$, then we get $l = 2m - 3$. If $V_{2m-4}(t) \leq 0$, then $V_{2m-5}(t) \geq 0$. So, there is a number l for which inequalities (8) take place.

From inequalities (8), we obtain that $V_1 = q_1 \frac{d}{dt} q_0 z >$, but q_0(t)=1 and $q_0(t) = 1$, which means $z'(t) > 0$,, since $z'_t = z'_r \cdot z'_t \geq 0$, then $z'_n(r) > 0$. So,

$$\left(\omega(r) r^{\frac{n-2m}{2}}\right)' = \left(\omega' + \frac{n-2m}{2} r^{-1} \omega\right) r^{\frac{n-2m}{2}} > 0$$

or

$$\omega' + \frac{n-2m}{2r} \omega > 0.$$

But

$$\omega' + \frac{n-2}{2r} \omega \geq \omega' + \frac{n-2m}{2r} \omega > 0$$

Therefore

$$\left(\omega\, r^{\frac{n-2}{2}}\right)' > 0.$$

Lemma 1 is proved.

Lemma 2. If $U(r) \geq 0$, then

$$\lim_{r \to \infty} \frac{u(r)}{\omega(r)} = 1.$$

Proof. The equation

$$F'' + \frac{1}{r} F' + \left(1 - \frac{(n-2)^2}{4r^2}\right) F = 0$$

It is a Bessel equation and has a fundamental system of solutions [10, p.185]

$$F_1 = \sqrt{\frac{2}{\pi r}} \sin(r - \mu) + O\left(r^{-\frac{3}{2}}\right),$$

$$F_2 = \sqrt{\frac{2}{\pi r}} \cos(r - \mu) + O\left(r^{-\frac{3}{2}}\right),$$

$$\mu = \frac{(n-2)\pi}{2} + \frac{\pi}{4}.$$

Then the Cauchy function $K(r,s)$ of equation (4) will have the form

$$K(r,s) = \frac{2}{\pi} \sqrt{\frac{1}{r \cdot s}} \sin(r - s) + \varphi(r,s),$$

$$\varphi(r,s) = O\left(r^{-\frac{3}{2}}\right) + O\left(s^{-\frac{3}{2}}\right).$$

According to the Cauchy formula, we have

$$F(t) = \sqrt{\frac{2}{\pi r}} \sin(r - r_1) + O\left(r^{-\frac{3}{2}}\right) + \frac{2}{\pi\sqrt{r}} \cdot \int_{r_1}^{r} s'^{-\frac{1}{2}} \sin(r - s)\, \omega_1(s) ds + \int_{r_1}^{r} \varphi(r,s)\, \omega_1(s) ds.$$

Consider the integral

$$I = \int_{r_1}^{r} s^{-\frac{1}{2}} \sin(r-s)\, \omega_1(s) ds.$$

Integrating in parts, we will have

$$I = \omega_1(r) r^{-\frac{1}{2}} - r_1^{-\frac{1}{2}} \omega_1(r_1) \cos(r-r_1) - \int_{r_1}^{r} \omega_1'(s) s^{-\frac{1}{2}} \cos(r-s)\, ds +$$

$$+\frac{1}{2} \int_{r_1}^{r} \omega_1(s) s^{-\frac{3}{2}} \cos(r-s)\, ds.$$

Next, we have

$$\left| \int_{r_1}^{r} \omega_1(s) s^{-\frac{3}{2}} \cos(r-s)\, ds \right| \le \omega_1(r) r^{-\frac{1}{2}}$$

$$\left| \int_{r_1}^{r} \omega_1'(s) s^{-\frac{1}{2}} \cos(r-s)\, ds \right| \le r^{\frac{1}{2}} |\omega_1(r) - \omega_1(r_1)|.$$

Thus

$$|I| = \left| \int_{r_1}^{r} \sin(r-s) \cdot \omega_1(s) ds \right| \le r^{-\frac{1}{2}} \omega_1(r) - r_1^{-\frac{1}{2}} \omega_1(r_1) \cos(r-r_1) +$$

$$+ r^{\frac{1}{2}} \omega_1(r) + r^{\frac{1}{2}} \omega_1(r_1) + r^{-\frac{1}{2}} \omega_1(r).$$

According to Lemma 1, we obtain

$$\left| \int_{r_0}^{r} \varphi(r,s) \omega_1(s) ds \right| \le K_1 \omega_1(r) r^{-\frac{1}{2}}, \quad K = const > 0.$$

Thus

$$|F(t)| \le \sqrt{\frac{2}{\pi r}} \sin(r-r_1) + O\left(r^{-\frac{3}{2}}\right) + r^{-1} \omega_1(r) + r_1^{-\frac{1}{2}} r^{-\frac{1}{2}} \omega_1(r_1) + \omega_1(r) +$$

Therefore, we finally have

$$\lim_{r\to\infty} \frac{|F(r)|}{|\omega_1(r)|} = 1,$$

since, $F(r) = U \cdot r^{\frac{n-2}{2}}$, $\omega_1(r) = \omega(r) r^{\frac{n-2}{2}}$, then

$$\lim_{r\to\infty} \frac{|U(r)|}{|\omega(r)|} = 1.$$

Lemma 2 is proved.

Now we turn to the consideration of inequality (2), which, after replacing $t = \ln r$ will take the form $(U \ge 0)$

$$L\omega + P(t) U^{\lambda} \exp(2mt) \le 0. \tag{9}$$

The fundamental system of solutions of the equation $L\omega = 0$ has the form:

a) if $n > 2m$, or n is any odd number, then

$$\omega_k(t) = \exp(2k-n), \qquad (k=1,2,\dots,m),$$
$$\omega_{m+k}(t) = \exp(2k-2)\,t, \qquad (k=1,2,\dots,m)$$

b) if $2 \le n \le 2m$ and n is an even number, then

$$\omega_k(t) = \exp(2k-n)\,t, \qquad \left(k=1,2,\dots,\frac{n}{2}\right);$$
$$\omega_{\frac{n}{2}+2k}(t) = \exp(2k), \qquad \left(k=1,2,\dots,m-\frac{n}{2}\right);$$
$$\omega_{\frac{n}{2}+2i+1} = t\exp(2it), \qquad \left(i=0,1,\dots,m-\frac{n}{2}\right);$$
$$\omega_{2m+1-\frac{n}{2}+i}(t) = \exp(2m-n+2i)\,t, \qquad \left(i=1,2,\dots,\frac{n}{2}-1\right).$$

To find the functions $q_k(t)$, first we calculate the determinants $W_k(t)$ $(k=1,2,\dots,2m)$
If $n > 2m$, or n is any odd number, then

$$W_k(t) = c_k \exp[k(1+k-n)t], \qquad (k=1,2,\dots,m)$$
$$W_{m+1}(t) = c_{m+1}\exp[m(m-n+1)t]$$
$$W_{m+k+1}(t) = c_{m+k+1}\exp\big[\big(m(m-n+1)+k(k+1)\big)t\big], \quad (k=1,2,\dots,m-1).$$

If n is even and 2≤n≤2m, then

$$W_k(t) = c_k\exp[k(k-n+1)t], \qquad \left(k=1,2,\dots,\frac{n}{2}-1\right),$$
$$W_{\frac{n}{2}}(t) = c_{\frac{n}{2}}\exp\left[-\frac{n(n-2)}{4}t\right], \qquad W_{\frac{n}{2}+1}(t) = c_{\frac{n}{2}+1}\exp\left[-\frac{n(n-2)}{4}t\right],$$
$$W_{\frac{n}{2}+2}(t) = c_{\frac{n}{2}+2}\exp\left[\left(-\frac{n(n-2)}{4}+2\right)t\right].$$

$$W_{\frac{n}{2}+2k+1}(t) = c_{\frac{n}{2}+2k+1}\exp\left[\left(-\frac{n(n-2)}{4}+2k(k+1)\right)t\right], \qquad \left(k=0,1,\dots,m-\frac{n}{2}+1\right),$$
$$W_{\frac{n}{2}+2k}(t) = c_{\frac{n}{2}+2k}\exp\left[\left(-\frac{n(n-2)}{2}+2k^2\right)t\right], \qquad \left(k=0,1,2,\dots,m-\frac{n}{2}+1\right),$$

$$W_{2m-\frac{n}{2}+k+1}(t) = c_{2m-\frac{n}{2}+k+1}\,\cdot$$
$$\cdot\exp\left[\left(-\frac{n(n-2)}{4}+(2m-n+2)\left(m-\frac{n}{2}\right)+k(2m-n+k+1)\right)t\right],$$
$$\left(1 \le k \le \frac{n}{2}-1\right).$$

Now substituting $W_k(t)$ into formula (6), we find

1) $q_0(t) = \exp[(n-2)t],\ q_i(t) = c_i\exp[(-2t)]\ (i=1,2,\dots,2m-1)$
$q_{2m}(t) = c_{2m}\exp[(2m-2)t]\ q_m(t) = c_m\exp[(2m-n)t],$

for $n > 2m$, or n is any odd number:

2) $q_0(t) = \exp[(n-2)t],\ q_i(t) = c_i\exp(-2t), \left(i=1,2,\dots,\frac{n}{2}-1,\frac{n}{2}+1\right)$
$q_{\frac{n}{2}+2k+1}(t) = c_{\frac{n}{2}+2k+1}\exp(-2t)\left(k=1,2,\dots,m-\frac{n}{2}\right)$

$$q_{\frac{n}{2}+2k}(t) = c_{\frac{n}{2}+2k}\left(k = 0, 1, 2, \dots, m - \frac{n}{2} + 1\right)$$
$$q_{2m-\frac{n}{2}+i+1}(t) = c_{2m-\frac{n}{2}+i+1}\exp(-2t), \left(i = 1, 2, \dots, \frac{n}{2} - 2\right)$$
$$q_{2m}(t) = c_{2m}\exp[(2m-2)t].$$

for $2 \leq n \leq 2m$ and n is an even number.

By virtue of the introduced notation (7), inequality (9) will take the form

$$V_{2m}(t) + P(t)U^{\lambda}\exp(2t) \leq 0 \qquad (10)$$

Before giving the main results, we present the following Lemma.

Lemma 3. If $U(t) \geq 0$ and $V_{2m}(t) \leq 0$, then there are numbers l and $t_l \geq t_0$ such that for $t \geq t_1$ inequalities (8) and

$$V_l(t) \leq 2^{l-1}(l-1)!\, V_l(t)[1 + o(1)]\exp[-2(l-1)t], \qquad (11)$$

if $n > 2m$ or n is any odd number and $l < m$..

$$V_l(t) \leq 2^p p!\,(n - 2m + 2p) \dots [n - 2m + 2(l-2)]\,(1 + o(1)V_1(t) \cdot$$
$$\cdot \exp[-(n - 2m + 2(l-2))t] \qquad (12)$$

if $l = m + p, p \geq 1, n > 2m$, or n is an odd number;

Proof. Let's prove inequality (11). The function $V_l(t)$ is non-negative and does not increase. We have

$$V_{l-1}(t) \geq \int_{t_0}^{t} V_l(s)\exp(2s)\,ds \geq \frac{1}{2}V_l(t)[1 + o(1)]\exp(2t)$$

$$V_{l-2}(t) \geq \frac{1}{2}\int_{t_0}^{t} V_l(t)[1 + o(1)]\exp(4s)\,ds \geq \frac{1}{2}\cdot\frac{1}{4}V_l(t)[1 + o(1)]\exp(4t)$$

. . . .

$$V_1(t) \geq \frac{1}{2^{l-1}(l-1)!}V_l(t)[1 + o(1)]\exp[2(l-1)t]$$

hence the inequality (11).

The proof of inequality (12) is similar to proof (11), only taking into account that

$$V_m(t) = \frac{dV_{m-1}}{dt}\exp[(2m-n)t].$$

If $U(x)$ is a non-oscillating solution of equation (1), then by definition it has no zeros in R_a^n for some $a > 0$ Therefore, we can consider $U(x) > 0$ в R_a^n. Then, according to Lemma 1, $\omega(r) > 0$ в R_a^n.

Theorem 1. If $\lambda > 1, n > 2m$, or $n = 2k + 1$ then for all solutions of equation (1) to fluctuate, it suffices that the conditions $a(x) \geq P(r) \geq 0$ and

$$\int_{r_0}^{\infty} P(r)r^{2m+n-\lambda(n-2)}dr = \infty. \qquad (13)$$

Proof. Let's assume that equation (1) has a non-oscillating solution $U(x)$. Then we can consider $U(x) \geq 0$ in $R_{r_0}^n$. For this solution, inequality (2) holds, which, after the above substitution $t = \ln r$ and the introduced notation (7), takes the form (10). Multiply inequality (10) by

$$(\omega\exp(n-2)\,t)^{-\lambda}\exp[(2m+n-4)t]$$

and integrate from t_0 to t :

$$\int_{t_0}^{t} \frac{V_{2m}(s)\exp[(2m+n-4)s]}{(\omega(s)\exp[(n-2)s])^{\lambda}}ds + \int_{t_0}^{t} P(s)\left(\frac{U}{\omega\exp(n-2)\,t}\right)^{\lambda}\exp[(2m+n-2)t]\,ds \le 0.$$

We integrate the first integral in parts, then applying Lemma 2 we get

$$\frac{\Phi(t)}{(\omega(t)\exp[(n-2)s])^{\lambda}} - \int_{t_0}^{t}\Phi(s)d[\omega(s)\exp[(n-2)s]^{-\lambda} + \tag{14}$$

$c = const$, where

$$\Phi(t) = V_{2m-1}(t)\exp[(n+2m-4)t] - (n+2m-4)V_{2m-2}(t)\exp[(n+2m-6)t] + \cdots + \tag{15}$$

$$Q(t) = (n+2m-4)(n+2m-6)\ldots(2l-2)V_l(t)\exp[2(l-1)t] \tag{16}$$

for $n > 2m$ either $n = 2k+1$, $(k \ge 1)$, and $l < m$,

$$\Phi(t) = V_{2m-1}(t)\exp[((n+2)m-4)t] - (n+2m-4)V_{2m-2}(t)\exp[(n+2m-6)t] + \cdots +$$
$$+(-1)^{i-1}(n+2m-4)(n+2m-6)\ldots\big(n+2m-4-2(i-2)\big)V_{2m-2}(t)\cdot$$
$$\cdot\exp[(n+2m-4-2(i-1)t)] + (n+2m-4)(n+2m-6)\ldots(n-2m+2l-2)\cdot$$
$$\cdot V_l(t)\exp[(n-2m+2l-2)t]. \tag{17}$$

$$Q(t) = (n+2m-4)(n+2m-6)\ldots(n-2m+2l-2)V_l(t)\exp[(n-2m+2l-2)t]. \tag{18}$$

For $l > m, l \ge 2m - \frac{n}{2} + 1, 2 \le n \le 2m$.

Inequality (7) implies that $\Phi(t) \ge 0$.. Consequently, from (14), by virtue of Lemma 3, we obtain

$$\int_{r_0}^{r} P(s)[1+o(1)]\exp[(2m+n-2)-\lambda(n-2)s]\,ds \le C + \int_{t_0}^{t}\frac{Q(s)ds}{[\omega(s)\exp(n-2)t]^{\lambda}},$$

where $c = const$ ($n > 2m$ or $n = 2k+1$).

If $l < m$, then according to equalities (15), (16) we have

$$\int_{t_0}^{t}\frac{Q(s)ds}{[\omega(s)\exp(n-2)s]^{\lambda}} = (n+2m-4)(2+2m-6)\ldots(2l-2)\int_{t_0}^{t}\frac{V_l(s)\exp(2ls)}{[\omega(s)\exp(n-2)s]^{\lambda}} \le$$
$$\le (n+2m-4)(n+2m-6)\ldots(2l-2)2^{l-1}(l-1)\int_{t_0}^{t}[1+o(1)]\frac{d[\omega(s)\exp(n-2)s]}{[\omega(s)\exp(n-2)s]^{\lambda}} =$$
$$= (n+2m-4)(n+2m-6)\ldots(2l-2)2^{l-1}(l-1)!\,[1+o(1)]\frac{1}{1-\lambda}\left[V_0^{1-\lambda}(t_0) - V_0^{1-\lambda}(t)\right].$$

where $V_0(t) = [\omega(t)\exp(n-2t)]$.

Since $\lambda > 1$ and $V_0(t)$ increases, it follows from the latter inequality and by virtue of Lemma 3 that

$$\int_{t_0}^{\infty}\frac{Q(t)dt}{[V_0(s)]^{\lambda}} < \infty,$$

Therefore

$$\int_{t_0}^{t} P(t)[1+o(1)]\exp[(2m+n-2)t - \lambda(n-2)t]\,dt < \infty.$$

From here, passing from t to r, we have

$$\int_{r_0}^{\infty} r^{2m+n-3-\lambda(n-2)} P(r)dr < \infty,$$

which contradicts condition (15) of the theorem.

In the case of $l > m$, it is treated similarly. In this case, instead of equalities (15), (16), equality (17), (18) should be used, respectively, and inequality (12) should also be used instead of inequality (11).

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