MATHEMATICAL SCIENCES

REFLECTIONS ON TEACHING THE METHOD OF CONSTANT VARIATION

Ma Lijun
School of Mathematics and statistics, JiNing Normal University, Ulanqab, 012000, Inner Mongolia, P. R. China
Scientific research project of JiNing Normal University, Project No.: jsky2024021
https://doi.org/10.5281/zenodo.15536087

Abstract

The method of constant variation is an effective method for solving linear differential equations. It is the result of Lagrange's eleven-year research, but we only use its conclusion without any process, which makes beginners feel mysterious and confused. In order to help students to understand the concept of the method of constant variation, and to stimulate them to think well, liberate their minds and be creative, this paper analyses and explores the method of constant variation.

Keywords: the method of constant variation; doubts; profiling

First-order linear ordinary differential equations, as one of the basics of ordinary differential equations, has a complete and systematic theoretical foundation and a rich practical background, which is of great significance to enhance students' motivation and thinking ability in learning the subsequent contents.Therefore, it is necessary to pay great attention to it in teaching. For first-order linear ordinary differential equations, the key lies in mastering its solution, which is usually introduced in the textbook is the method of constant variation, which has clear ideas, concise and ingenious derivation, and it is easy for teachers and students to grasp the basic steps.However, most of the students were confused about the method of constant variation in which a constant c is transformed into an unknown function $c(x)$, which was not explained in the textbook.Therefore, on the premise of elaborating the theoretical basis of the method of constant variation, this paper explores the solution method from the structural characteristics of the equation itself and the general solution of the corresponding chi-square linear differential equations in response to the problems of the students in learning the content of this section. Students' ability to raise, analyse and solve problems is cultivated from different perspectives, and the teaching effect of eliminating doubts and knowing the causes and consequences of the results is achieved, which greatly promotes the expansion of students' thinking ability.

1 Introduction to the method of constant variation

First-order non homogeneous linear ordinary differential equation

$$y' + P(x)y = Q(x) \tag{1}$$

where $P(x), Q(x)$ is a continuous function with respect to x.

When $Q(x) \equiv 0$, the equation

$$y' + P(x)y = 0 \tag{2}$$

is a homogeneous linear differential equation.

Firstly, the general solution of the corresponding homogeneous linear differential equation (2) is solved, and the general solution of equation (2) can be obtained by applying the method of separated variables as

$$y = ce^{-\int P(x)dx} \tag{3}$$

where c is an arbitrary constant.

Step 2 Let $y = u(x)e^{-\int P(x)dx}$ (changing any constant c into an unknown function $u(x)$) be the solution of equation (1), and substituting into (1), we get

$$u(x) = \int Q(x)e^{\int P(x)dx}dx + d,$$

where d is an arbitrary constant.

Step 3 Substituting $u(x) = \int Q(x)e^{\int P(x)dx}dx + d$ back gives the general solution of equation (1) as

$$y = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx + d\right),$$

where d is an arbitrary constant.

2 The Idea of the method of constant variation

The above is the method of constant variation introduced in the current university textbook on ordinary differential equations, which is a practical and effective method for solving non homogeneous linear ordinary differential equation with clear ideas, compact and concise derivation, easy to be grasped by both teachers and students, and clear to be applied.However, when students learn the method of constant variation, they will have the following doubts: Why do we need to transform non homogeneous linear ordinary differential equation into homogeneous linear ordinary differential equation to solve them? And how do you think of turning an arbitrary constant c into a function to be determined $u(x)$? The textbook does not provide a detailed explanation. Therefore, it is necessary to discuss the thinking process of the method of constant variation.

2.1 Find the general solution of equation (1) from the characteristics of the equation itself

Method 1 As with the familiar solution of the separation of variables equation, equation (1) is deformed into

$$\mathrm{d}y=\left(Q(x)-P(x)y\right)\mathrm{d}x\text{,or}$$

$$\frac{\mathrm{d}y}{y}=\left(\frac{Q(x)}{y}-P(x)\right)\mathrm{d}x,$$

Integrating both sides gives

$$\ln|y|=\int\frac{Q(x)}{y}dx-\int P(x)\mathrm{d}x+c',$$

where c' is an arbitrary constant.

Since y is a function of x, note that $\varphi(x)=\int\frac{Q(x)}{y}dx$, where $\varphi(x)$ is the unknown function, substituting into the above equation yields $y=ce^{\varphi(x)}e^{-\int P(x)dx}$, Denote by $u(x)=ce^{\varphi(x)}$,then $y=u(x)e^{-\int P(x)dx}$,where $u(x)$ is the function to be found.The second step of the method of constant variation is then used to find the general solution of equation (1) as

$$y=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+d\right),$$

where d is an arbitrary constant.

Method 2 Let $y=u(x)v(x)$,where both $u(x)$ and $v(x)$ are functions with respect to x. Substituting $y=u(x)v(x)$ into equation (1), the collation yields

$$u'(x)v(x)+u(x)\left[v'(x)+v(x)P(x)\right]=Q(x)\text{,(4)}$$

if we now want to solve for $u(x)$ by the method of separation of variables, then we need to make $v'(x)+v(x)P(x)=0$, separate the variables and integrate, which gives us

$$v(x)=c_1e^{-\int P(x)\mathrm{d}x}\text{, (5)}$$

where c_1 is an arbitrary constant.

The result $v(x)$ has now been solved, and we proceed to solve for $u(x)$. Substituting equation (5) into equation (4), we get

$$c_1\frac{\mathrm{d}u}{\mathrm{d}x}e^{-\int P(x)\mathrm{d}x}=Q(x),$$

continuing to separate variables and integrate, we have

$$u(x)=\frac{1}{c_1}\int e^{\int P(x)\mathrm{d}x}Q(x)\mathrm{d}x+c_2,$$

consequently

$$y=u(x)v(x)=\left(\frac{1}{c_1}\int e^{\int P(x)\mathrm{d}x}Q(x)\mathrm{d}x+c_2\right)c_1e^{-\int P(x)\mathrm{d}x}=\left(\int e^{\int P(x)\mathrm{d}x}Q(x)\mathrm{d}x+c\right)e^{-\int P(x)\mathrm{d}x},$$

(where $c=c_1\cdot c_2$).

Further observation shows that solving the differential equation $v'(x)+P(x)v(x)=0$ containing $v(x)$ is essentially solving the general solution of the homogeneous linear ordinary differential equation $y'+yP(x)=0$. Solve for $y=\mathrm{c}e^{-\int P(x)\mathrm{d}x}$, where c is an arbitrary constant.

Substituting $y=u(x)e^{-\int P(x)\mathrm{d}x}$ into equation (1) solve for $u'(x)e^{-\int P(x)\mathrm{d}x}=Q(x)$.Solve for $u(x)$ and substitute for $y=u(x)e^{-\int P(x)\mathrm{d}x}$ to get the answer

$$y=e^{-\int P(x)\mathrm{d}x}\left(\int Q(x)e^{\int P(x)\mathrm{d}x}\mathrm{d}x+c\right).$$

Method 3 Inspired by the Leibniz formula

$$\frac{d}{dx}u(x)v(x)=v(x)\frac{du(x)}{dx}+u(x)\frac{dv(x)}{dx},$$

multiplying both sides of equation (1) by $\varphi(x)$ ($\varphi(x)\neq 0$), we have

$$\varphi(x)\frac{dy}{dx}+\varphi(x)P(x)y=\varphi(x)Q(x).$$

How to choose $\varphi(x)$,such that $\frac{d\varphi(x)}{dx}=\varphi(x)P(x)$, then the above equation can be

$$\frac{d\left(\varphi(x)y\right)}{dx}=\varphi(x)Q(x),$$

integrating both sides gives $\varphi(x)y=\int\varphi(x)Q(x)dx+c$, where c is an arbitrary constant.Integrate the two sides of $\frac{d\varphi(x)}{dx}=\varphi(x)P(x)$ to get $\varphi(x)=ce^{\int P(x)dx}$, and substitute back into the above equation to get

$$y=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+k\right).$$

2.2 Solving the general solution of equation (1) from the general solution of homogeneous linear ordinary differential equation

Observation of equations (1) and (2) reveals that equation (2) is a special case of equation (1), in that they agree at the left end and differ at the right end. Therefore, it is conceivable that there is a correlation but also a difference between their solutions.An attempt is made to solve the general solution of equation (1) by means of the general solution (3) of equation (2). Obviously, this is not true if equation (3) is substituted directly into equation (1).For equation (1) to be always equal, there must be an additional function on x on the left side of the equation corresponding to $Q(x)$ on the right side. According to the formula for the derivative of the product of functions, It is necessary to transform equation (3), and then the derivation of the equation gives an additional function on x.

Observing the expression of equation (3), it can be found that equation (3) consists of two parts: constant c and exponential function $e^{-\int P(x)dx}$. To make equation (3) a function on x with an extra term after derivation, it can be transformed as follows.

The function $u(x)$ $\left(u(x)\neq 0\right)$ with respect to x is introduced by the four fundamental operations at different positions of equation (3), i.e., the following transformations are available:

$$y=ce^{-\int P(x)dx}+u(x) \tag{6}$$

$$y=ce^{-\int P(x)dx}-u(x) \tag{7}$$

$$y=cu(x)e^{-\int P(x)dx} \tag{8}$$

$$y=\frac{c}{u(x)}e^{-\int P(x)dx} \tag{9}$$

$$y=\left(c+u(x)\right)e^{-\int P(x)dx} \tag{10}$$

$$y=\left(c-u(x)\right)e^{-\int P(x)dx} \tag{11}$$

$$y=ce^{-\int P(x)dx+u(x)} \quad (12) \qquad y=ce^{-\int P(x)dx-u(x)} \quad (13)$$

$$y=ce^{-u(x)\int P(x)dx} \quad (14) \qquad y=ce^{\frac{-\int p(x)dx}{u(x)}} \quad (15)$$

where c is an arbitrary constant.

Substituting equation (6) into equation (1), we have

$$u'(x)=P(x)u(x)+Q(x),$$

this equation is also a non homogeneous linear ordinary differential equation, which is cyclic and difficult to solve $u(x)$. Therefore, it is not possible to find the solution of equation (1) by transforming equation (6). Equation (7) is similar to equation (6), so the solution of equation (1) cannot be obtained by transforming equation (7).

Substituting equation (8) into equation (1), we have $u'(x)=\dfrac{Q(x)e^{\int P(x)dx}}{c}$, and integrating both sides we have

$$u(x)=\int\frac{Q(x)e^{\int P(x)dx}}{c}dx+c_1,$$

substituting $u(x)$ into equation (8), let $cc_1=c'$, then $y=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+c'\right)$ is the general solution of equation (1), where c' is an arbitrary constant.

Substituting equation (9) into equation (1), we have $\dfrac{u'(x)}{u^2(x)}=-\dfrac{Q(x)e^{\int P(x)dx}}{c}$,integrating both sides we have

$$\frac{1}{u(x)}=\int\frac{Q(x)e^{\int P(x)dx}}{c}dx+c_1,$$

substituting $\dfrac{1}{u(x)}$ into equation (9), let $cc_1=c'$, then $y=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+c'\right)$ is the general solution of equation (1), where c' is an arbitrary constant.

Substituting equation (10) into equation (1), we have $u'(x)=Q(x)e^{\int P(x)dx}$, integrating both sides we have

$$u(x)=\int Q(x)e^{\int P(x)dx}dx+c_1,$$

substituting $u(x)$ into equation (10), let $c+c_1=c'$, then $y=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+c'\right)$ is the general solution of equation (1), where c' is an arbitrary constant. Equation (11) is similar to equation (10), and by the same token, the solution of equation (1) can be found by transforming equation (11).

Substituting equation (12) into equation (1), we have $u'(x)e^{u(x)}=\dfrac{Q(x)e^{\int P(x)dx}}{c}$, integrating both sides we have

$$e^{u(x)}=\int\frac{Q(x)e^{\int P(x)dx}}{c}dx+c_1,$$

substituting $e^{u(x)}$ into equation (12), let $cc_1=c'$, then $y=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+c'\right)$ is the general solution of equation (1), where c' is an arbitrary constant. Equation (13) is similar to equation (12), and by the same token, the solution of equation (1) can be found by transforming equation (13).

Substituting equation (14) into equation (1), we have

$$\frac{dce^{-u(x)\int P(x)dx}}{dx}=-cP(x)e^{-u(x)\int P(x)dx}+Q(x),$$

let $\varphi(x)=e^{-u(x)\int P(x)dx}$, then

$$\frac{dc\varphi(x)}{dx}=-cP(x)\varphi(x)+Q(x),$$

this equation is a non homogeneous linear ordinary differential equation, so the form of the required equation appears cyclically, therefore, the solution of equation (1) can not be found by transforming equation (14). Similarly, the case of equation (15) is similar to that of equation (14) and is omitted here.

Comprehensive analysis shows that among the 10 different deformations, only 6 transformations (8), (9), (10), (11), (12) and (13) can find the general solution of equation (1). The 6 transformations are carefully analysed, and it is found that all of them can be reduced to transforming any constant c in equation (3) into an unknown function $u(x)$, and then substituting it into equation (1) to find out $u(x)$, and then to find out the general solution of equation (1).

3 Teaching Summary

Teachers in teaching this section of the content, can not be limited to the textbook analysis of the explanation, need to start from the characteristics of the equation itself, with the help of students familiar with the knowledge, for students to carry out detailed in-depth analysis, and to carry out specific derivation, layer by layer questioning, and gradually cultivate the students to find, analyse and problem-solving skills.At the same time, students in the study of the content of the section, should not be a passive recipient of knowledge, but should be actively and enthusiastically into the learning, encountered there are doubts and uncertainty, should take the initiative to ask questions, with questions, follow the teacher's thinking, layer by layer, in-depth, personal deduction. Only in this way, we can achieve the elimination of doubt, not only know the truth but also know the reason for the effectiveness of teaching.

References:

1. Wang Guoxiong,Zhou Zhiming,Zhu Siming. Ordinary differential equations[M]. Beijing:Higher Education Press,2020:26-27.
2. Kuang Yuyang,Li Xinghua,Wang Tairong. Study on the method of constant variation for solving differential equations[J]. Yangzhou Vocational University,2023,27(03):47-50.
3. Kuang Yuyang, Li Xinghua, Wang Tairong. the method of constant variation teaching research[J]. Journal of Yangling Institute of Technology,2023,22;15-19.
4. Wang Yanhua and Pan Zhigang, “Exploration of the idea of the method of constant variation and its application in low-order variable coefficient non chi-square differential equations” was published in “Science and Technology Wind”, 2022, pp. 90-92.
5. Yang F. A pedagogical discussion on the method of constant variation[J]. Journal of Higher Education Science,2010,30(02):92-95.