MATHEMATICAL SCIENCES

RESEARCH ON THE APPLICATION OF FUNCTION MONOTONICITY IN SOLVING HIGH SCHOOL MATHEMATICS PROBLEMS

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Abstract

As a core mathematical concept describing the interdependent relationship between variables, function monotonicity is crucial for studying the changing trends of functions. Not only serving as the cornerstone of calculus, it also acts as a key tool for solving complex problems in high school mathematics. This paper systematically sorts out and explores the multi-dimensional applications of function monotonicity in solving high school mathematics problems. Specific application areas include: determining the maximum and minimum values of functions using monotonicity, judging the number of real roots of equations, solving complex problems involving parameter ranges, proving inequalities, and its transformed application in sequence problems. Through the systematic analysis of various application examples, this study aims to reveal the value of function monotonicity in improving the efficiency and accuracy of problem-solving, and it holds significant educational significance for cultivating students' logical reasoning ability and abstract thinking.

Keywords: Function Monotonicity; Derivatives; Maximum and Minimum Value Problems; Real Roots of Equations; Parameter Ranges; Inequalities; Sequences

1 Introduction

As one of the fundamental concepts in mathematics, functions serve as a powerful tool for describing the interdependent relationships between two variables in the real world. In practical scenarios, understanding the changing trends of quantities is essential—for instance, the variation of speed with time in travel problems and the fluctuation of profit with sales volume in business problems. Within the development of mathematics itself, in-depth exploration of the changing patterns of function values as the independent variable varies, along with the intuitive manifestation of "rising" or "falling" function graphs, has driven the emergence of the concept of monotonicity. In scientific fields, concepts such as motion velocity and energy change in physics are closely related to function monotonicity. From the perspective of mathematics education, function monotonicity not only lays the foundation for calculus but also acts as a key tool for solving problems involving equations, inequalities, and sequences. Therefore, a thorough understanding of the judgment methods and applications of function monotonicity is of great significance for cultivating students' logical reasoning abilities and abstract thinking.

In recent years, with the advancement of the reform in high school mathematics curriculum, the requirements for students' core mathematical competencies have been continuously elevated. The learning of function monotonicity serves as a crucial carrier for cultivating students' core competencies such as mathematical abstraction, logical reasoning, and mathematical modeling. Particularly after the introduction of derivatives as a powerful tool for determining function monotonicity, the scope of applications of monotonicity in solving complex problems has been greatly expanded. However, in the practice of high school mathematics teaching, students often struggle to flexibly apply monotonicity, a theoretical tool, to various types of problems.

Based on this, this paper focuses on exploring the multi-dimensional applications of function monotonicity in solving high school mathematics problems. Specifically, through typical examples, this paper will conduct a detailed analysis of the application strategies, problem-solving steps, and thinking methods of function monotonicity in five aspects: solving maximum and minimum value problems, determining the number of real roots of equations, finding parameter ranges, proving inequalities, and judging sequence properties. Through such systematic summary and induction, this study aims to provide clearer thinking guidance for high school mathematics teaching, help students master the mathematical transformation thinking of "converting abstraction into concreteness and complexity into simplicity", and thereby significantly improve the efficiency and accuracy of problem-solving.

2 Application of Function Monotonicity

In high school mathematics teaching, the learning of function monotonicity holds multiple educational values. Firstly, it helps students grasp the inherent laws of mathematical changes and serves as a cornerstone for understanding other function-related concepts. Through the dual cognition of the analytical expressions and graphs of the definition and concept of monotonicity, abstract mathematical principles can be visualized, thereby establishing intuitive mathematical thinking. This cognitive approach not only deepens the understanding of basic knowledge such as linear functions and quadratic functions but also cultivates the ability to solve comprehensive problems (e.g., maximum and minimum value problems, equation solving, and parameter range analysis) using monotonicity. Through systematic training on monotonicity-related problems, students can gradually master the mathematical transformation thinking of "converting abstraction into concreteness and complexity into simplicity," and such thinking training significantly improves the efficiency and accuracy of problem-solving.

2.1 Application in Solving Maximum and Minimum Value Problems

Typically, the maximum (minimum) value of a function can be determined once its monotonicity is known. The introduction of the concepts of maximum/minimum values and extreme values in textbooks not only improves the knowledge system but also emphasizes key points. Firstly, the geometric intuition of a function's maximum (minimum) value is reflected by the ordinate of the highest (lowest) point on its graph. Secondly, the "point" in the terms "maximum point" and "minimum point" (as defined) does not refer to a point represented by a two-dimensional ordered pair, but rather the value of the independent variable corresponding to the function's maximum or minimum value, which can be understood as a point on a one-dimensional number line. Additionally, it is important to note that a function may have multiple maximum or minimum points.

Example 1: Find the maximum and minimum values of function $f\left(x\right)=x^3-3x$ on interval $\left[-2,2\right]$.

Solution: Differentiating function $f\left(x\right)$ gives $f'\left(x\right)=3x^2-3$.

Let $f'\left(x\right)=0$, i.e., $3x^2-3=0$, and solve for $x=\pm 1$. When $x\in\left(-2,-1\right)$, $f'\left(x\right)>0$, and the function $f\left(x\right)$ is monotonically increasing at this time. When $x\in\left(-1,1\right)$, $f'\left(x\right)<0$, and the function $f\left(x\right)$ is monotonically decreasing at this time. When $x\in\left(1,2\right)$, $f'\left(x\right)>0$, and the function $f\left(x\right)$ is monotonically increasing at this time. Calculate the endpoint values and extreme values, we have:

$$f\left(-2\right)=-2,\ f\left(-1\right)=2,\ f\left(1\right)=-2,$$
$$f\left(2\right)=2.$$

Therefore, the maximum value of function $f\left(x\right)=x^3-3x$ on interval $\left[-2,2\right]$ is 2, and the minimum value is -2.

When solving the maximum and minimum values of a function using its monotonicity, the steps are as follows: first, differentiate function $f\left(x\right)$ to obtain $f'\left(x\right)$; then, set $f'\left(x\right)=0$ to find critical points or points where the derivative does not exist; divide the domain using these critical points and points of non-existent derivative; next, determine the monotonicity of the function; Then calculate the function values at the interval endpoints, points where the derivative does not exist, and critical points respectively; finally, obtain the maximum and minimum values through comparison. This method organically combines the monotonic characteristics of the function with extreme value determination, which not only reflects the rigor of mathematics but also provides a practical problem-solving approach.

2.2 Application in Determining the Number of Real Roots of an Equation

By analyzing the monotonicity of a function, the number of its real roots can be determined.

Example 2: Determine the number of real roots of Equation $x^3-3x+1=0$ on interval $\left(-2,2\right)$.

Solution: Let $f\left(x\right)=x^3-3x+1$, differentiating function $f\left(x\right)$ gives $f'\left(x\right)=3x^2-3$. Set $f'\left(x\right)=0$, i.e., $3x^2-3=0$, and solve for the variable to obtain $x=\pm 1$. When $x\in\left(-2,-1\right)$, $f'\left(x\right)>0$, and function $f\left(x\right)$ is monotonically increasing at this time. When $x\in\left(-1,1\right)$, $f'\left(x\right)<0$, and the function $f\left(x\right)$ is monotonically decreasing at this time. When $x\in\left(1,2\right)$, $f'\left(x\right)>0$, and the function $f\left(x\right)$ is monotonically increasing at this time. Therefore, on interval $\left(-2,2\right)$, $f\left(-1\right)$ is the maximum value and $f\left(1\right)$ is the minimum value. Calculate the function values at the endpoints and critical points,

$$f\left(-2\right)=-1,\ f\left(-1\right)=3,\ f\left(1\right)=-1,\ f\left(2\right)=3.$$

Since $f\left(-2\right)<0$, $f\left(-1\right)>0$, there is one zero point in $\left(-2,-1\right)$; furthermore, $f\left(1\right)<0$, $f\left(-1\right)>0$, so there is one zero point in $\left(-1,1\right)$; and as $f\left(1\right)<0$, $f\left(2\right)>0$, there is one zero point in $\left(1,2\right)$.

In summary, Equation $x^3-3x+1=0$ has three real roots in interval $\left(-2,2\right)$.

When applying the monotonicity of a function to determine the number of real roots of an equation, the steps are as follows: first, differentiate function $f\left(x\right)$ to obtain $f'\left(x\right)$; then, set $f'\left(x\right)=0$ to find the critical points; divide the domain into monotonic intervals using these critical points and determine the monotonicity of function $f\left(x\right)$; finally, the number of real roots of the equation can be derived based on the number of zeros of the function.

2.3 Application in Determining the Range of Parameters

The monotonicity of a function plays an important role in solving problems related to determining the range of parameters, and the method of separating parameters needs to be flexibly applied. If function $f(x)$ is increasing on interval (a,b), then $f'(x)\geq 0$ holds constantly on interval (a,b). If function $f(x)$ is decreasing on interval (a,b), then $f'(x)\leq 0$ holds constantly on interval (a,b).

Example 3: Given function

$$f(x)=\begin{cases}-\dfrac{1}{4x-4}, x\leq\dfrac{3}{4}\\ \log_a(4x)-1, x>\dfrac{3}{4}\end{cases}\qquad (*)$$

is monotonic on R, find the range of the real number a.

Solution: According to the problem statement, when $x\leq\frac{3}{4}$, $f(x)=-\frac{1}{4x-4}$. Differentiating function $f(x)$ gives $f'(x)=\frac{4}{(4x-4)^2}>0$, and function $f(x)$ is monotonically increasing on interval $\left(-\infty,\frac{3}{4}\right]$ at this time. Since function (*) is monotonic on R, when $x>\frac{3}{4}$, function $f(x)=\log_a(4x)-1$ must be monotonically increasing on $\left[\frac{3}{4},+\infty\right)$. Differentiating function $f(x)$ gives $f'(x)=\frac{4}{x\ln a}$. If function $f(x)$ is monotonically increasing, then $f'(x)>0$ holds constantly on interval $\left[\frac{3}{4},+\infty\right)$; therefore, $\ln a>0$, i.e., $a>1$. Obviously, function (*) is monotonically increasing on interval $(-\infty,+\infty)$; therefore, $\log_a\left(4\times\frac{3}{4}\right)-1\geq-\frac{1}{4\times\frac{3}{4}-4}$, and solving for the variable gives $1<a\leq\sqrt{3}$.

Here, a piecewise function is taken as an example. When applying the monotonicity of a function to find the range of parameters, first analyze the monotonicity of each segment of the piecewise function separately; then, the overall monotonicity of the function can be derived based on the monotonicity of each segment; finally, solve for the parameter range by comparing the values of each segment of the function.

2.4 Application in Inequalities

Let function $f(x)$ be continuous on its domain interval I and differentiable within I. If $f'(x)>0$ holds within I, then function $f(x)$ is monotonically increasing on I. If $f'(x)<0$ holds within I, then function $f(x)$ is monotonically decreasing on I.

Example 4: Prove that when $x>0$, $\ln(x+1)>x-\frac{1}{2}x^2$.

Proof: Let

$$f(x)=\ln(x+1)-\left(x-\frac{1}{2}x^2\right)=\ln(1+x)-x+\frac{1}{2}x^2.$$

The domain of function $f(x)$ is $(-1,+\infty)$. Differentiating function $f(x)$ gives $f'(x)=\frac{x^2}{x+1}$. When $x\in(-1,+\infty)$, $f'(x)>0$; therefore, function $f(x)$ is monotonically increasing on $(-1,+\infty)$. Therefore, when $x>0$, $f(x)>f(0)=0$; that is, when $x>0$, $\ln(x+1)>x-\frac{1}{2}x^2$.

When solving inequality problems using the monotonicity of a function, first construct a new function. After determining the domain of the new function, differentiate it to obtain the monotonicity and sign of the new function, thereby judging whether the inequality holds.

2.5 Application in Sequences

A function is a fundamental concept in mathematics that describes the relationship between variables. A sequence can be regarded as a discrete function defined on the set of positive integers (or a finite subset of the positive integers). The regularity of a sequence precisely reflects the corresponding relationship between the independent variable n and the function value a_n of the underlying function. Approaching sequence teaching from a functional perspective helps students quickly establish connections between knowledge related to sequences. Therefore, the relevant theories and methods of function monotonicity can be ingeniously applied to solving sequence problems.

Example 5: Given that sequence $\{a_n\}$ satisfies $a_n=\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{3n+1}, n\in N^*$, if $a_n>2b-5$ holds constantly, find the maximum value of b (where b is a natural number).

Solution: Since

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots\cdots + \frac{1}{3n+1}.$$

$$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots\cdots + \frac{1}{3(n+1)+1}.$$

thus

$$a_{n+1} - a_n = \frac{2}{(3n+2)(3n+3)(3n+4)}.$$

From $n \in N^*$, we obtain $a_{n+1} - a_n > 0$, i.e., $a_{n+1} > a_n$; therefore, $\{a_n\}$ is an increasing sequence, with the first term a_1 being the minimum value and $a_1 = \frac{13}{12}$. To ensure $a_n > 2b-5$ holds constantly, it is necessary that $a_1 > 2b-5$; solving this gives $b < \frac{73}{24}$, so the maximum value of b is 3.

When solving sequence problems using the monotonicity of a function, we can first analyze the monotonicity of the sequence through the difference method: regard $a_{n+1} - a_n$ as a function. If the sequence corresponding to this function is monotonically increasing, the minimum value of the sequence occurs at the first term; if the sequence is monotonically decreasing, the maximum value of the sequence occurs at the first term.

3 Conclusion

The research on function monotonicity not only possesses profound theoretical value but also plays an irreplaceable and important role in solving practical problems and cultivating students' logical thinking and application abilities. This paper systematically sorts out the application examples of function monotonicity in core high school mathematics fields such as maximum and minimum value problems, real roots of equations, solving parameter ranges, inequality proofs, and sequence properties, comprehensively revealing its multi-dimensional and practical application value. By providing clear problem-solving steps and thinking guidance for various types of problems, these methods can effectively help students quickly select and implement reasonable problem-solving strategies according to the characteristics of specific questions, thereby significantly improving the speed and accuracy of problem-solving.

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