SINGLE-MODE VORTEX AS AN ELEMENTARY VORTEX GEOMETRIC STRUCTURE OF AN IDEAL FLUID

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Abstract

A single-mode vortex is considered as an elementary solution to the equations of an ideal incompressible fluid in Arnold's geometric formulation [1]. It is shown to be defined by a single harmonic mode and to represent a divergence-free velocity field with helical streamline geometry. In the Lie algebra of volume-preserving diffeomorphisms, a single-mode vortex corresponds to an Abelian element whose commutator with itself is zero, ensuring the absence of nonlinear self-generation of modes and the stationarity of the solution [2]. It is established that such vortices are geodesic trajectories on the group of diffeomorphisms and serve as fundamental building blocks of more complex vortex structures.

Keywords: vortical dynamics, topological methods in hydrodynamics, single-mode vortex, kern, algebra of Lie, diffeomorphism, helical geometry.

1. General Definition. In the theory of an ideal incompressible fluid, particularly in Arnold's geometric formulation [1], elementary vortex structures called single-mode vortices play a fundamental role. A single-mode vortex represents a velocity field corresponding to a single harmonic mode in the Fourier expansion. In its most general form, such a field can be described using a scalar stream function.

$$\psi(x) = A\,Cos(k \cdot x), \quad (1)$$

where $x \in \mathbb{R}^n$ is the spatial coordinate, $k \in \mathbb{R}^n$ is the fixed wave vector, A is the amplitude, and $k \cdot x$ denotes the scalar product. The corresponding velocity field is defined through the oblique gradient operator

$$v = \nabla^{\perp}\psi \text{ , (2)}$$

Which guarantees the fulfillment of the incompressibility condition $\nabla \cdot v$. In the two-dimensional case, this means

$$v(x) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right) = Ak^{\perp}\,Sin(k \cdot x), \quad (3)$$

where $k^{\perp} = (-k_2, k_1)$ is the vector orthogonal to k.

Geometrically, a single-mode vortex is characterized by the fact that all of its dynamic properties are determined by a single spatial harmonic. Lines of constant phase $k \cdot x = Const$ form a regular family of surfaces along which the fluid moves. In cylindrical coordinates, these lines take the form of helical curves, and the corresponding vortex structure realizes helical geometry. In this sense, a single-mode vortex represents the simplest helical structure possessing complete spatial regularity.

From the perspective of Arnold's geometric mechanics [1], a single-mode vortex has a particularly important algebraic property. The space of all admissible velocity fields of an ideal fluid can be viewed as an infinite-dimensional Lie algebra of divergence-free vector fields. In this space, the elementary harmonics $e_k = e^{i(k \cdot x)}$ form a natural basis, and the corresponding vortex fields can be expressed in terms of these modes. The Lie commutator of two modes is defined by the relation

$$[e_k, e_\ell] = (k \wedge \ell)e_{k+\ell}\ , \quad (4)$$

where $(k \wedge \ell)$ is the oriented area of the parallelogram constructed on the vectors k and ℓ. In particular, for a single-mode vortex the following holds:

$$[e_k, e_k] = 0\ . \quad (5)$$

This means that a single-mode vortex does not interact with itself and does not generate new modes. Dynamically, this corresponds to the velocity field maintaining its structure over time and not undergoing nonlinear deformation. Thus, a single-mode vortex is a stationary solution to the Euler equations and represents a geodesic line in the group of volume-preserving diffeomorphisms.

The vortex field corresponding to a single-mode structure also has a simple form. Vorticity is defined as the velocity curl:

$$\omega = \nabla \times v\ . \quad (6)$$

Substituting the expression for velocity shows that the vorticity is also proportional to the same harmonic:

$$\omega \propto Cos\,(k \cdot x). \quad (6^*)$$

This means that the vortex is completely described by a single spatial mode and does not contain more complex spatial scales.

Geometrically, a single-mode vortex corresponds to a minimal helical surface, in particular a helicoid. Such a surface can be parameterized as

$$X(r,\theta) = (r\,Cos\,\theta, r\,Sin\,\theta, a\theta), \quad (7)$$

where the parameter a is determined by the components of the wave vector. The radial coordinate remains constant along the streamlines, and the fluid moves strictly along helical trajectories. This reflects the fundamental regularity of the single-mode structure [2-4].

From a physical perspective, a single-mode vortex is an elementary vortex configuration that contains no internal interaction between different spatial scales. Increasingly complex vortex structures can be obtained as a superposition of several single-mode vortices. It is the interaction of different modes that leads to radial transport, geometric deformation, and the formation of more complex surfaces such as catenoids and spherical structures. However, the single-mode vortex remains the fundamental building block of vortex dynamics, defining the simplest stable state of vortex motion.

Thus, a single-mode vortex represents an elementary divergent-free velocity field corresponding to a single harmonic mode, possessing helical geometry and being a stationary solution to the equations of ideal hydrodynamics. Its fundamental role is that it serves as the basic element from whose interactions increasingly complex vortex structures emerge.

2. Single-mode vortex in the context of the Lie algebra of diffeomorphisms. The formalization of a single-mode vortex naturally arises within the framework of the geometric theory of ideal fluids, where the space of all admissible motions is considered as a group of volume-preserving diffeomorphisms of the flow domain. Let M be a smooth oriented manifold with a fixed volume shape. Then the group of all smooth volume-preserving diffeomorphisms $\mathrm{Diff}_\mu(M)$, forms an infinite-dimensional Lie group. The corresponding Lie algebra $\mathfrak{diff}_\mu(M)$ consists of all divergence-free vector fields on M, that is, those velocity fields $v(x)$ for which the incompressibility condition holds.

$$\nabla \cdot v = 0. \qquad (8)$$

The algebraic structure of this Lie algebra is given by the commutator of vector fields, defined as

$$[u, v] = (u \cdot \nabla)\, v - (v \cdot \nabla) u\,. (9)$$

In this formulation, the Euler equation for an ideal fluid acquires a particularly simple geometric meaning: it describes geodesic motion on the group $\mathrm{Diff}_\mu(M)$ with respect to a right-invariant metric induced by kinetic energy. The fluid velocity vector at each instant is an element of the Lie algebra $\mathfrak{diff}_\mu(M)$, and its evolution is determined by the intrinsic geometry of this group. Thus, the dynamics of vortices can be completely described in terms of the Lie algebra of divergence-free diffeomorphisms.

In the case of a periodic domain, for example, a two-dimensional torus T^2, the natural basis of the Lie algebra is the harmonic modes of the form

$$e_k(x) = e^{i(k \cdot x)}\,, \qquad (10)$$

where k is an integer wave vector. These modes form a complete basis in which any divergence-free field can be expanded in a Fourier series. The commutator of two such modes has the form (4), where the quantity

$$k \wedge \ell = k_1 \ell_2 - k_2 \ell_1 \qquad (11)$$

represents the oriented area of the parallelogram constructed on the vectors k and ℓ. This relation is fundamental, since it shows that the interaction of two vortex modes leads to the emergence of a new mode corresponding to the sum of the wave vectors.

A single-mode vortex corresponds to the choice of a single basis element of the Lie algebra. Let the velocity field be defined by a single mode e_k. Then the corresponding vector field has the form (3) and belongs to the Lie algebra $\mathfrak{diff}_\mu(M)$. Since the commutator of this mode with itself is zero (5), a single-mode vortex is an Abelian element of the Lie algebra [2]. This means that the corresponding velocity field does not generate new modes during its own evolution and retains its structure over time. Geometrically, this corresponds to the fact that a single-mode vortex defines a one-dimensional Lie subalgebra, which generates a one-parameter subgroup of diffeomorphisms. This subgroup describes the helical motion of fluid particles.

In cylindrical coordinates, such motion takes the form of a helical trajectory:

$$z = a\theta\,, \qquad (12)$$

where the parameter a is determined by the components of the wave vector. Thus, a single-mode vortex realizes the simplest geometric structure, corresponding to a one-parameter subgroup of the group of volume-preserving diffeomorphisms. Its stability is a direct consequence of the fact that it corresponds to a geodesic line on the group $\mathrm{Diff}_\mu(M)$ [2].

If the velocity field contains more than one mode, the corresponding element of the Lie algebra ceases to be Abelian [2-4]. The commutators of the different modes become non-zero, and a nonlinear interaction arises, leading to the generation of new modes. This corresponds to going beyond the one-dimensional subalgebra and transitioning to a more complex geometry of motion. In this sense, a single-mode vortex is an elementary geometric object in the Lie algebra of diffeomorphisms, serving as a fundamental building block for more complex vortex structures.

Thus, the Lie algebra of volume-preserving diffeomorphisms provides a natural and rigorous mathematical framework for describing vortices. A single-mode vortex is an elementary element of this algebra, corresponding to one harmonic mode and one geodesic trajectory in the diffeomorphism group. Its geometric simplicity and stability make it a fundamental object, the interactions of which give rise to increasingly complex vortex and geometric structures [2-4].

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