PHYSICAL SCIENCES

ON THE ROLE OF THE "LEMNISCATIC CORE" IN LAME ELLIPSOIDAL COORDINATES

Milyute E.
IRG "LITAVEM-3", Vilnius, Lithuania
https://doi.org/10.5281/zenodo.18840584

Abstract

The functions introduced by Gabriel Lamé in connection with the distribution of heat in a homogeneous ellipsoid [1-2], which have since been given Lamé's name, have become the subject of a large number of studies. The works of Liouville and Heine [3-4], independent but simultaneous, are the first in history and among the most important. Then come the excellent studies of Charles Hermite, continued by various authors on the integration of the differential equation satisfied by these functions. Closer to them are two separate groups of significant works: F. Klein [5-6] and his students generalize the Lamé functions and consider them in a completely new way.

Keywords: ellipsoidal vortex, lemniscatic kernel, Lame functions of the first and second kinds, conformal mapping, Lame ellipsoidal coordinates, lemniscatic strip, confocal (homofocal) surfaces, potential theory.

1. History of the issue

The functions introduced by Gabriel Lamé in connection with the distribution of heat in a homogeneous ellipsoid [1-2], which have since been named after Lamé, have become the subject of a large number of studies. The works of Liouville and Heine [4, 3], independent but simultaneous, are the first in history and among the most important. Then come the excellent studies of Charles Hermite, continued by various authors on the integration of the differential equation satisfied by these functions. Closer to them are two separate groups of significant works: F. Klein [5-6] and his students generalize Lamé's functions and examine them in a completely new way.

Following the remarkable introduction of a short treatise by Carl Neumann, a follower of Heine in potential theory, "On the Expansion of a Function with Imaginary Argument" (1862) [7], German mathematicians in the late 19th century developed an interest in Lamé functions for solving conformal problems. In his work "Development of Functions of a Complex Variable from Lamé Functions for Given Spherical Functions" (1882) [8], F. Lindemann laid the foundations for a simple solution of "conformal" problems by means of a transition from Lamé functions to spherical functions, without resorting to complex theoretical calculations. His student O.T. Volk (1925) proved the validity of using the Lame function (21) for an imaginary quantity for a conformal mapping [9,10] (in the formula of Lindemann and Volk, $\boldsymbol{c}$ was replaced by $\boldsymbol{a}$; since at that time due attention was not paid to the precise interpretation of quantities and their belonging to the generally accepted quantities in the calculus of an ellipse) and the research was continued by Volk's student O.E. Stanaitis (1937, 1939), who studied the integral properties of the Lame functions [11].

Based on the above, we can study in detail the mechanism of circulating vortex flows on layered (on different layers) surfaces of an ellipsoidal vortex using Lame ellipsoidal coordinates.

Therefore, there is very little information left in modern mathematics about such special Lamé functions; we can further very briefly dwell on some proofs of well-known Lamé functions, which can be found in this article.

2. Spherical Laplace function (3R).

We have a differential equation:

$$\frac{1}{Sin\,\theta}\frac{d^2f}{d\varphi^2}+\frac{1}{Sin\theta}\frac{d}{d\theta}\left(Sin\theta\frac{df}{d\theta}\right)+n(n+1)f=0, \quad (1)$$

where n is a positive integer. Solutions regular on the surface of a sphere are called spherical Laplace functions [8]. For each value of n, as is known, there exist $2n+1$ functions $f_{mn}(m=1,2,3,\dots,2n+1)$, which are expressed as follows:

$P_{nm}(Cos\theta)\ Cos\ m\varphi,\ P_{nm}(Cos\theta)\ Sin\ m\varphi$, (2)

where $m=0,1,2,\dots,n$, and $P_{nm}(Cos\theta)$ are Legendre functions of the n-th order. Next, we assume that in the region $-\pi\le\varphi\le\pi$, $0\le\theta\le\pi$ these functions are orthogonal and generally unrelated.

Introducing elliptic coordinates [11]:

$$\begin{cases} x=\ Cos\theta=\frac{\mu\nu}{bc} \\ y=Sin\theta\ Cos\varphi=\frac{\sqrt{\mu^2-b^2}\sqrt{b^2-\nu^2}}{b\sqrt{c^2-b^2}} \\ z=Sin\ \theta Sin\varphi=\frac{\sqrt{c^2-\mu^2}\sqrt{c^2-\nu^2}}{c\sqrt{c^2-b^2}} \end{cases} \quad (3)$$

Then equation (1) changes to another form:

$$\frac{d^2f}{du^2}+\frac{d^2f}{dv^2}+n(n+1)(\mu^2-\nu^2)f=0, \quad (4)$$

Where the values of u and υ denote:

$$u=\int_b^{\mu}\frac{d\mu}{\sqrt{\mu^2-b^2}\sqrt{c^2-\mu^2}}\ ,\upsilon=\int_b^{\mu}\frac{d\nu}{\sqrt{b^2-\nu^2}\sqrt{c^2-\nu^2}} \quad (5)$$

Here μ and ν vary within the interval:

$$b\le\mu\le c\ ,-b\le\nu\le b,0<b<c$$

3. The "lemniscatic kernel" of the Lamé function of the first kind and its harmonic analysis

When searching for solutions to the differential equation (1), we will use a special Lamé formula that connects two Lame functions $E_1(\mu)$ and $E_2(\nu)$ of two arguments μ and ν :

$$f=E_1(\mu)E_2(\nu) \quad (6)$$

And we find the Lamé functions $E_1(\mu)$ and $E_2(\nu)$ of the quantities $z,\sqrt{z^2-b^2},\sqrt{z^2-c^2}$ satisfy the same differential equation $E_{ns}(z)$ – Lamé functions of the first kind of degree $n\ at\ z=\mu=\nu$

$(z^2-b^2)(z^2-c^2)\frac{d^2E_{nm}}{dz^2}+z(2z^2-b^2-c^2)\frac{dE_{nm}}{dz}+$
$+\left((b^2+c^2)\,v_s^n-\,n(n+1)z^2\right)E_{nm}=0$, (7)

where v_s^n is the root of some algebraic equation:

$v_s^n \sim \alpha_1 n^2+\beta_1 n+\gamma_1=\alpha_1(n+\,0.5)^2+$
$+(\beta_1-\alpha_1)+\gamma-0.25\,\alpha_1$, (8)

where $0\le\alpha_1\le\,1, b, c$ are real numbers. From here we have that for odd $n\;\Longrightarrow\;n\to\sigma=n/2$, for even $n\;\Longrightarrow\;n\to\sigma=(n-1)/2$.

Thus, the function (7) has $s=\sigma+1$ functions,

$K_{ns}(z)=a_0z^n+a_1z^{n-2}+\cdots$ (9)

As a solution, where the coefficients $a_1, a_2, \dots$ are determined by integration using the infinite series method. To determine the coefficients of the class $K_{ns}(z)$ of functions E_{ns}, we first find them from the system:

$$\begin{cases} 2(2n-1)a_1=p(K-n^2)a_0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 2m(2n+1-2m)a_m=p\left[K-\left((n+2-2m)\right)^2\right]a_{m-1}+ \\ +\,q(n+4-2m)(n+3-2m)a_{m-2}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{\sigma+1}=0=\left[K-\left((n-2\sigma)\right)^2\right]a_0+ \\ q(n+4-2\sigma)(n+1-2\sigma)a_{\sigma-1}\,, (10) \end{cases}$$

Where the notations p, q, K mean:

$p=b^2+c^2, q=b^2c^2,\; K=v_s^n$ (11)

Eliminating the coefficients $a_1, a_2, \dots, a_\sigma$ from the system of equations (10), we obtain an algebraic equation of degree $\sigma+1$, which defines K. It has been proven that this algebraic equation has $s+1$ distinct real roots. From this and therefore follows that there exist $\sigma+1$ functions $K(z)$, that are not linearly related. Therefore, each individual case will always have its own formula, which may not be applicable to another case. Lamé functions possess this unique property, so they cannot be precisely defined for use as a general formula for other cases.

Then, upon substitution, we have

$t=\int_c^z\frac{\sqrt{(b^2+c^2)\alpha_1-z^2}}{(z^2-b^2)(z^2-c^2)}dz\,, y=\frac{u}{\sqrt[4]{(b^2+c^2)\alpha_1-z^2}}$ (12)

then equation (7), which will have the form

$\frac{d^2u}{dt^2}+\left[\left(n+\frac{1}{2}\right)^2+\frac{z^2+4B(\beta_1+\gamma_1)}{4A}+\frac{z^4-b^2c^2}{2A^2}-\right.$
$\left.-\,\frac{5z^2(z^2-b^2)(z^2-c^2)}{4A^3}\right]u=0$, (13)

where $B=b^2+c^2$, $A=\left((b^2+c^2)\alpha_1-z^2\right)$.

So we have a final solution

$e^{it}=\zeta\; eps\left(\int_c^z\frac{z-\sqrt{z^2-\alpha_1B}}{\sqrt{(z^2-b^2)(z^2-c^2)}}dz\right)=\zeta\; eps\;\mathfrak{S}$ (14)

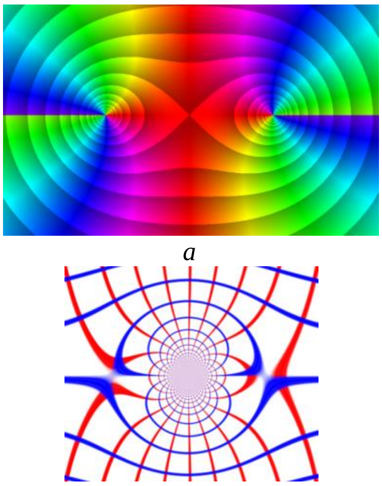
from here we obtain the "*lemniscatic kernel*" function for the Lamé integral:

$\zeta=\frac{\sqrt{z^2-b^2}-\sqrt{z^2-c^2}}{\sqrt{c^2-b^2}}=eps\left(-\int\,\frac{zdz}{(z^2-b^2)(z^2-c^2)}\right)$ (15)

In the conformal mapping, function (15) for $|\zeta|=\rho$ $(0\;\le\rho\;\le1$ or $1\;\le\rho\le\;\infty)$ yields a system of curves that are very similar in system to the *sofocal curves* of the lemniscate and the sofocal curves of the Cassini ovals [8-10, 18]. Formula (15) gives a system of annular rings connected through the center, as shown in Fig.1.1 and 1.2.

From Fig.1.1a, it is clear that at the conformal mapping of function (15) there are two cuts here, and the semi-hyperbolas pass from one quadrant through the cuts to the other quadrant strictly according to the inversion law. Here we see a *focal point* C_1, which is connected through the cuts to *another focal point* C_2.

We can denote this visual imaginary connection by a special function that will connect both the conformal relation and the spherical one through the center of coordinates.



a

b

Fig.1.1 a) Conformal mapping of a function (26) [1,18]; b) The $\mathfrak{S}$ function is surrounded by a blue circle in the center. This function is described in more detail in the paper [18].

$f(x,y,z)=E(\mu^2)E(\nu^2)$, (16)

where

$\Delta_2F=0$ (17)

And since function (15) has two discontinuities, we will assume that in the Riemann variation we are dealing with a two-sheeted Riemann surface, and this surface is doubly connected, as shown in Fig. 1.2 from the work [18].

From this it is clear that the curves $|e^{i\zeta}| = \rho$ cannot intersect with the curves $|\zeta| = \rho$, since the function $\mathfrak{S}$ has a non-zero real part (Fig. 1.2) and it affects other isogonal lines that fall within the region of the kernel of the function $\mathfrak{S}$.

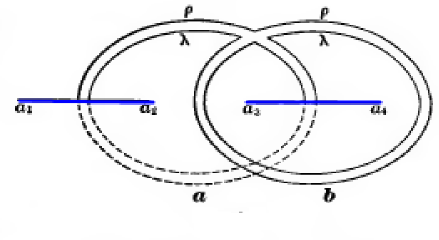


Fig.1.2 Doubly connected Riemann surface of function (15) according to Lindemann [18].

4. Lamé functions of the second kind.

We obtain, as usual, by Euler's method, the second solution of the Lamé equation: this function, known as the Lamé function of the second kind and important in potential theory, is defined by the expression

$$\boldsymbol{E^n(\rho) = (2n + 1)R^n(\rho) \int_{-\infty}^{\rho} \frac{\rho d\rho}{[R^n(\rho)]^2\sqrt{(\rho^2 - a^2)(\rho^2 - b^2)(\rho^2 - c^2)}}} \quad (18)$$

or with a variable *u*

$$\boldsymbol{E^n(z) = (2n + 1)R^n(z) \int_0^u \frac{du}{[R^n(z)]^2}} \quad (19)$$

When z → $\mathbf{0}$ or $\boldsymbol{\rho} \to \infty$, principal value $\boldsymbol{E^n = \rho^{-n-1}}$, and the value $\boldsymbol{R^n = \rho^n}$.

The introduction and study of this second solution are due to Liouville and Heine [3-4]. Various authors have also dealt with it; F. Lindemann [8] developed the quantity $\boldsymbol{(z_1 - z)^{-1}}$, where $\boldsymbol{z}$ and $\boldsymbol{z_1}$ are any two real quantities successively occurring in products of the type $\boldsymbol{R^n(z)E^n(z_1)}$; he also studies other expansions of the same kind in series of Lamé functions of the first or second kind: the regions of convergence of these various series are, in general, bounded by curves of the fourth order.

The fundamental role played in the problem of equilibrium of a rotating fluid by equations involving the product $\boldsymbol{R^nE^n}$ has prompted authors studying the problem to seek a form for the function $\boldsymbol{E}$ that is characteristic of a numerical solution. Lyapunov [12-14] presents expressions involving symmetric functions of the roots of the functions $\boldsymbol{R}$, or more precisely, the polynomial $\boldsymbol{f(\rho^2)}$ adjacent to these functions: these formulas are still quite complex.

But if we take into account the *principle of orthogonality* and the *principle of chirality*, then the formula of orthogonal identity (formula (20) from work [18]) is solved using the law of mapping of an ellipsoidal vortex (see Chapter 6 from work [18]) or using the law of symmetry of the lemniscatic strip.

$$\boldsymbol{\frac{X+iY}{r-Z} = \frac{r+Z}{X-iY} = \mathbb{Z} = Const} \quad (20)$$

Thanks to this law of symmetry and the mapping of the lemniscatic strip, the problem is solved more easily

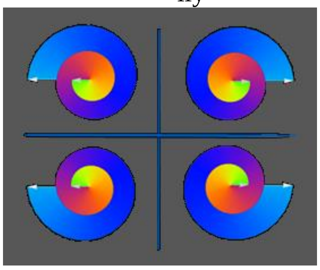


Fig. 1.3 Mirror and inversion symmetry with respect to the OX and OY axes of the simply connected Riemann model (helical spiral)

Riemann's two-sheet geometry (see Fig. 2.3 from work [18] and Fig.1.3). With such an inversion mapping, the flow direction along the lemniscatic strip is preserved; this is even more clearly seen in conformal geometry.

This phenomenon must often be taken into account when describing the distribution of "discharged" matter in the interior of an ellipsoidal closed vortex. Here, in this case, we can easily transition from spherical (elliptical) trigonometry to hyperbolic geometry by equating the spherical radius to the imaginary unit [15-16, 18].

Each function $\boldsymbol{E(z)}$ can also be used as an entire function $\boldsymbol{E(\zeta)}$, where from the system of equations (22-28) we can find the function for the "*lemniscatic core*" for ellipsoidal vortex:

$$\boldsymbol{\zeta = \frac{\sqrt{z^2-b^2}-\sqrt{z^2-c^2}}{\sqrt{c^2-b^2}}} \quad (21)$$

5. The relationship of Lamé homofocal surfaces with elliptical coordinates of field potentials in the 3R system and their characteristics.

Heine himself noted that it was very difficult to solve spatial physical problems of heat and potential in three dimensions without a suitable coordinate system. It was only after Lamé introduced special constructions that simplified the approach to studying three known geometric figures simultaneously: the ellipsoid, the hyperboloid of one sheet, and the hyperboloid of two sheets. Before Lamé, scientists used polar coordinates on a spherical (ellipsoidal) surface, which at the time was computationally cumbersome.

Let us rewrite the system of these figures "according to Lame" in the words of Heine [3]:

Let there be a system of surfaces subject to the conditions

$\boldsymbol{mx^2 + ny^2 + lz^2} = 1$, (22)

where $\boldsymbol{m} = \frac{1}{\lambda^2}$, $\boldsymbol{n} = \frac{1}{\lambda^2 - b^2}$, $\boldsymbol{l} = \frac{1}{\lambda^2 - c^2}$

and $\boldsymbol{\lambda > c}$, $\boldsymbol{c > \lambda > b, b > \lambda > 0}$

than

$\frac{x^2}{\rho^2} + \frac{y^2}{\rho^2 - b^2} + \frac{z^2}{\rho^2 - c^2} = \mathbf{1}$ Ellipsoid

$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - b^2} - \frac{z^2}{c^2 - \mu^2} = \mathbf{1}$ One-sheet (23) hyperboloid

$\frac{x^2}{\nu^2} - \frac{y^2}{\rho^2 - \nu^2} - \frac{z^2}{c^2 - \nu^2} = \mathbf{1}$ Two-sheeted hyperboloid

Such surfaces are usually called *confocal* in German works, and Lame called them *homofocal* (Lame, p. 156) [2].

From system (23) we find the values for *x, y, z* for elliptical coordinates:

$$\boldsymbol{x} = \frac{\boldsymbol{\rho\mu\nu}}{\boldsymbol{bc}}$$

$$\boldsymbol{y} = \frac{\sqrt{\rho^2 - b^2}\sqrt{\mu^2 - b^2}\sqrt{b^2 - \nu^2}}{b\sqrt{c^2 - b^2}} \quad (24)$$

$$\boldsymbol{z} = \frac{\sqrt{\rho^2 - c^2}\sqrt{c^2 - \mu^2}\sqrt{c^2 - \nu^2}}{c\sqrt{c^2 - b^2}}$$

From here we find expressions for the quantities:

$\boldsymbol{\xi = x - \sqrt{x^2 - 1}}$ (25)

$\boldsymbol{\eta} = \frac{\sqrt{\mu^2 - b^2} - i\sqrt{c^2 - \mu^2}}{\sqrt{c^2 - b^2}}$ (26)

And additional formulas for finding the relationship between two independent coordinate systems:

$$\begin{cases} 2\sqrt{\mu^2 - b^2} = \sqrt{c^2 - b^2}\,(\eta^{-1} + \eta) \\ 2\sqrt{\mu^2 - c^2} = \sqrt{c^2 - b^2}\,(\eta^{-1} - \eta) \end{cases} \quad (27)$$

$$\begin{cases} 2\sqrt{\mu^2 - b^2} = \sqrt{c^2 - b^2}\; Cos\,\varphi \\ 2\sqrt{c^2 - \mu^2} = \sqrt{c^2 - b^2}\; Sin\,\varphi \end{cases}$$

from the system of equations (27) we find

$\boldsymbol{(27) \;\Rightarrow\; \eta = Cos\,\varphi - i\,Sin\,\varphi}$ (28)

The most striking thing is that Lamé anticipated the ease of calculating such complex three-dimensional structures. At the time Lamé published his work on isogonal lines and surfaces, few scientists paid attention to the importance of Lamé's work, but I note that Kelvin took these works with him to England in order to understand the mathematical formulas that he later applied in developing the stability of the rotation of the ellipsoid and in order to be able to apply these conical coordinates. His two-volume treatise on philosophy (on physics) [17], jointly with Tait, had a great influence on the direction of theoretical physics at that time in Europe and in Russia as a whole. Chebyshev himself repeatedly spoke about the problem of the stability of the ellipsoid (a problem raised by Kelvin) [12]. In my case, the ellipsoidal vortex, due to its unique core structure (Fig. 5.2 from [18]), allows its emerging internal vortex domains (flows) to rotate according to the geropoloid (herpoloid) law. These vortex flows in the closed system of the ellipsoidal vortex are characterized by a very high level of self-organization and a clear hierarchy in motion. Here, the vortex motion (not to be confused with the rotational type of solenoidal motion) has many analogs in nature of manifestations with semi-lemniscate motions, which are clearly observed in some cases when equations (29) and (30) are satisfied.

$\boldsymbol{\xi} = \frac{\sqrt{x^2 - b^2}}{\sqrt{a^2 - b^2}}$, $\boldsymbol{\eta} = \frac{\sqrt{a^2 - b^2}}{\sqrt{a^2 - b^2}}$ (29)

$$\boldsymbol{\omega = u + iv} = \frac{\sqrt{x^2 - b^2} - \sqrt{x^2 - c^2}}{\sqrt{c^2 - b^2}} \quad (30)$$

For $\boldsymbol{b = 2}$ and $\boldsymbol{c = 1}$, clear signs of a doubly connected three-field (six-field) lemniscate are observed [18]

6. Conclusions

Based on the above, we can conclude that the theory of Lamé functions forms a comprehensive mathematical foundation for studying problems of potential, heat conduction, and spatial conformal transformations in ellipsoidal coordinates. Historical development, beginning with the work of Gabriel Lamé and continued by Joseph Liouville and Eduard Heine, then Neumann, Lindemann, and his successor, Otto Volk, has shown that these functions possess unique analytical properties and require an individual approach in each specific case.

The relationship between the spherical Laplace functions and Lamé's ellipsoidal coordinates allows us to move from the standard spherical description to a system of second-order homofocal surfaces - the ellipsoid and hyperboloids. This representation significantly simplifies the solution of spatial problems and reveals the internal structure of solutions through the separation of variables and the construction of functions - integrals of the first and second kind. The introduction of a "*lemniscatic kernel*" and analysis of the corresponding conformal mappings demonstrate that solutions to the Lamé equation are naturally interpreted in the geometry of *lemniscatic strips* [18], significantly simplifying the study of the mathematical model of a two-sheeted Riemann surface. This emphasizes the importance of the principles of orthogonality, symmetry, and analytic continuation in the study of these functions.

Geometric interpretation through *homofocal surfaces* and *ellipsoidal coordinates* makes it possible to apply the obtained results to the modeling of complex domain-like closed vortex structures, where ordered and hierarchical dynamics of internal flows are observed.

Thus, Lamé functions act as a link between analytical methods, geometric constructions, and physical vortex models, providing a universal framework for describing complex spatial closed vortex processes.

References:

1. E. Miliūtė. (2023) Development of functions of a complex variable on the Lamé functions in conformal geometry. *64-oji Lietuvos matematikų draugijos konferencija,* Vilnius, Lietuva

2. G.Lamé. (1837) *J. Liouville* **2**, 247; (1839) **4**, 126; (1869) Leçons sur les Coordonnées Curvilignes. Paris, Mallet-Bachelier

3. E. Heine. (1864) Handbuch der Kugelfunktionen, erste Auflage.

4. J. Liouville. Lettres sur diverses questions

d'analyse et de physique mathématique concernant l'ellipsoïde, adressées à M.P.H.Blanchet, *Journ. de Math*. T. **XI**, P. 261; *Liouville Journ*. T. **IV**

5. F. Klein (1922) Gesammelte mathematische Abhand-lungen. Band 2, Berlin

6. F. Klein (1881) Über Lamésche Funktionen. *Math.Ann*. Bd. **19** :237-246

7. C. Newmann. (1862) Ueber die Entwicklung einen Funktion mit imaginären Argument nach den Kugelfunktionen, erster und zweiter Art, Halle

8. F. Lindemann. (1882) Entwieklung der Funktionen einer complexen Variabeln nach Laméschen Funktionen und nach Zugeordenten der Kugelfunktionen, *Mathematische Annalen*, vol. **19**.

9. O.T.Volk. (1925) Ueber die Entwicklung von Funktionen zweier komplexen Veränderlichen nach Lamèschen Funktionen, *Mathematische Zeitschrift*, Band **23**, PP. 224-237

10. O.T.Volk. (1925) Die Abbildung $\zeta = \left(\sqrt{z^2-b^2}-\sqrt{z^2-c^2}\right)/\sqrt{c^2-b^2}$ **,** *MGF Darbai*, T.**2**, PP: 34-50

11. O.E.Stanaitis. (1937) Iš Lamė funkcijų teorijos. *MGB Darbai*, T **XI**, I sąsiuvinis, PP. 2-24

12. А.Л.Цыкало. (1988) Александр Михайлович Ляпунов (1857-1918), М.: Наука

13. А.М.Ляпунов. (1885) Теория потенциала

14. А.М.Ляпунов. (1954) Собрание сочинений. т. 1, Изд-во Акад. наук ССР, Москва

15. N.J. Lobatschefskij. (1902) Pangeometrie. 1856 (übersetzt und herausgegeben von H.Liebmann), Leipzig

16. H. Liebmann. (1923) Nichteuklidische Geometrie, Walter de Gruyter & Co., Berlin

17. W.*Thomson, P.G.Tait. (1867) Treatise on Natural Philosophy. Oxford.* 2nd edition, 1883

18. E. Milyute (2023) Investigation of Conformal Mappings in Solving Problems of Vortex Dynamics.