MATHEMATICAL SCIENCES

ON THE GEOMETRY OF DEVELOPABLE SURFACES IN STRATIFIED INCOMING VORTICAL FLOW OVER A CONE

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Abstract

The paper considers a mathematical model of the formation of a developing surface in a stratified oncoming vortex flow when interacting with a conical obstacle. Similar structures arise in the hydrodynamics of swirling flows, atmospheric vortex systems, as well as in a number of technical applications related to turbomachines and aerodynamic devices. It is shown that the surface of the vortex layer descending from the cone can be described as a spiral ruled surface, the geometry of which is determined simultaneously by the shape of the obstacle and the structure of the oncoming flow. For the description, a system of equations of a stratified fluid in the Boussinesq approximation is used. In cylindrical coordinates, a parameterization of the surface is introduced through the radius function $r(\theta, z, t)$, depending on the angular coordinate, vertical coordinate and time. Expressions are obtained for the surface normal, coefficients of the first and second quadratic forms, Gaussian and mean curvature. The condition for the developability of the surface is derived and the criterion for the separation of the vortex layer from the surface of the cone is formulated. It is shown that the horizontal sections of the surface under consideration have the shape of an Archimedes spiral, and the pitch of the spiral is determined by the ratio of the layer descent velocity to the local angular velocity of the flow. The resulting model allows us to relate the geometry of the resulting surface with the parameters of the oncoming stratified vortex flow and the cone angle

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Keywords: oncoming stratified flow, matter transfer, vortex dynamics of substance, circulation, twist, turbulence, vortex layer separation criterion, cone cavity, developable surface, Gaussian curvature, vortex-sheet separation line, torse.

1. Introduction

Swirling stratified flows are widely encountered both in natural phenomena and in engineering problems [4-8]. Examples of such structures are atmospheric vortices, water funnels, flows in reactors, vortex structures in turbomachines and vortex separation devices. One of the characteristic features of such flows is the formation of complex vortex surfaces and spiral layers that arise when the flow interacts with geometric obstacles.

Of particular interest is the case of interaction of a swirling flow with axisymmetric bodies. If the oncoming flow has both axial velocity and azimuthal swirl, then when it flows around bodies of rotation, the formation of spiral vortex surfaces is possible. Such surfaces can come off the edge of an obstacle and spread in space in the form of spiral shells.

A developable surface is understood as a surface with zero Gaussian curvature. Such surfaces can be turned flat without stretching or compressing. Classic examples include cylindrical surfaces, cones and helicoids. However, in hydrodynamic problems, more complex surfaces may arise that retain local linearity and approach developable surfaces.

The purpose of this work is to construct a mathematical model that allows us to describe the geometry of such a surface and relate its shape to the flow parameters and the geometry of the obstacle. In particular, it is necessary to determine:

- structure of the oncoming stratified vortex flow;
- geometric description of the surface of the vortex layer;
- normal and curvature of the surface;
- condition of developability of a surface;
- criterion for the separation of the vortex layer from the surface of the cone.

2. Basic equations of stratified flow

To solve the problem under consideration, we introduce the function

$$v = (ur, u\theta, uz) \quad (1)$$

in cylindrical coordinates (r, θ, z).

The basic equations of motion of a stratified fluid in the Boussinesq approximation have the form

$$\begin{gathered} \nabla \cdot \mathbf{v} = 0, \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla p + b\,\mathbf{e}_z, \\ \frac{\partial b}{\partial t} + (\mathbf{v} \cdot \nabla) b = 0. \end{gathered} \quad (2)$$

Here p is pressure, ρ_0 is reference density, b is buoyancy. The evolution of stratification is described by the transport equation

$$\frac{\partial b}{\partial t} + (\mathbf{v} \cdot \nabla) b = 0. \tag{3}$$

For a stable stratified liquid, you can take

$$b = -N^2 z \tag{4}$$

where N is Brunt–Väisälä frequency, b is buoyancy.

3. Basic field of the oncoming vortex flow

The incoming vortex flow is assumed to be axisymmetric. Its structure is defined by the distributions of the axial and azimuthal velocities. We employ a convenient model of the axisymmetric flow upstream of the cone.

$$u_z = U(z),$$

$$u_\theta = \Omega(z)\, r \quad \text{или} \quad u_\theta = \frac{\Gamma(z)}{2\pi r}\left(1 - e^{-r^2/a^2}\right), \tag{5}$$

$$u_r \approx 0.$$

where $U(z)$ is the incoming axial velocity, $\Omega(z)$ is the local angular velocity, and Γ is the circulation.

To satisfy the incompressibility condition, a weak radial component is introduced.

$$u_r = -\frac{r}{2}\frac{\partial U}{\partial z}. \tag{6}$$

Then

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0. \tag{7}$$

4. Geometry of a Conical Obstacle

When such a flow interacts with a conical obstacle, a vortex surface $F(r, \theta, z, t) = 0$ is formed, issuing from the surface of the cone. The geometry of the obstacle in the form of a cone is described by the radius function

$$r_c\,(z) = r_0 + (z - z_0)\tan\alpha\,, \tag{8}$$

where α is the cone half-angle, $z_{\,0}$ is the level where the cone begins, and r_0 is the radius at the cross-section $z = z_0$. In Cartesian coordinates, the surface of the cone is defined by the equation:

$$x^2 + y^2 = r_c^2\,(z).$$

In cylindrical coordinates (r, θ, z), it can be written as

$$u_z = U(z,t), \qquad u_\theta = \Omega(z,t)\, r, \qquad u_r = -\frac{r}{2}\,\partial_z U(z,t), \tag{9}$$

with buoyancy b (4).

The normal to this surface can then be expressed as

$$F_c(r,z) = r - r_c(z) = 0, \quad \nabla F_c = \boldsymbol{e}_r - \tan\alpha\, \boldsymbol{e}_z \tag{10}$$

To describe the resulting surface, a cylindrical function is introduced.

$$r(\theta, z, t) = r_c(z) + a(z,t)\big(\theta - \theta_0(z,t)\big). \tag{11}$$

This formula defines a helical surface, which in Cartesian coordinates can be written in parametric form.

$$\mathbf{X}(\theta, z, t) = (r(\theta, z, t)cos\theta, r(\theta, z, t)sin\theta, z). \tag{12}$$

Horizontal cross-sections of this surface have the form of an Archimedean spiral. The pitch of the spiral is determined by the function

$$a(z,t) = \frac{V_s(z,t)}{\Omega(z,t)}. \tag{13}$$

Here $V_s\,(z,t)$ is the velocity at which the vortex sheet leaves the surface of the cone. In the simplest model, it is proportional to the axial flow velocity.

$$V_s(z,t) = \mu\, U(z,t) Sin\,\alpha\,. \tag{14}$$

Thus, the main equation of the surface is obtained

$$r(\theta, z, t) = r_0 + (z - z_0)\tan\alpha + \mu\frac{U(z,t)\sin\alpha}{\Omega(z,t)}(\theta - \theta_0(z,t)). \tag{15}$$

Further analysis of the surface requires the computation of its geometric characteristics. The derivatives of the radius function take the form

$$\begin{aligned} r_\theta &= a(z,t), \\ r_z &= \tan\alpha + a_z(\theta - \theta_0) - a\,\theta_{0z}. \end{aligned} \tag{16}$$

Thus, we obtain an implicitly developable helical surface, for which the Riemann helicoid is a particular case:

$$\boxed{F(r,\theta,z,t) = r - r_c(z) - a(z,t)\big(\theta - \theta_0(z,t)\big) = 0} \tag{17}$$

where r_c determines the geometry of the cone, $\theta_0\,(z,t)$ is the phase line of the vortex-sheet shedding, $\alpha(z,t)$ is the local pitch of the spiral unfolding, and μ is the coupling coefficient between the flow and the cone surface.

5. The condition that the surface is a material surface If the surface is transported by the flow, it must satisfy the transport equation.:

$$\frac{\partial F}{\partial t}+u_r\frac{\partial F}{\partial r}+\frac{u_\theta}{r}\frac{\partial F}{\partial \theta}+u_z\frac{\partial F}{\partial z}=0. \tag{19}$$

For our form, we have

$$F_r=1,\qquad F_\theta=-a(z,t),\qquad F_z=-r_c'(z)-a_z(\theta-\theta_0)+a\,\theta_{0z},$$
$$F_t=-a_t(\theta-\theta_0)+a\,\theta_{0t}. \tag{20}$$

Thus, the complete equation takes the form:

$$-a_t(\theta-\theta_0)+a\theta_{0t}+u_r-\frac{u_\theta}{r}a+$$
$$+u_z\left(-r_c'(z)-a_z(\theta-\theta_0)+a\theta_{0z}\right)=0. \tag{21}$$

Since

$r_c'(z)=\tan\alpha$, (22)

we obtain a closed-form PDE governing the evolution of the spiral developable surface:

$$-a_t(\theta-\theta_0)+a\theta_{0t}+u_r-a\,\frac{u_\theta}{r}+$$
$$+u_z\left(-r_c'(z)-a_z(\theta-\theta_0)+a\theta_{0z}\right)=0. \tag{23}$$

6. Developability Condition for the Parametric Representation of the Surface

If we wish to determine a ruled developable surface, using (12–13) we can write

$$\mathbf{X}(s,\lambda,t)=\mathbf{C}(s,t)+\lambda\,\mathbf{d}(s,t), \tag{24}$$

subject to the **zero Gaussian curvature condition**, which leads to **the unfolding of a stratified vortex shell** shed from the conical obstacle

$$\boxed{\det(\mathbf{C}_s,\mathbf{d},\mathbf{d}_s)=0.} \tag{25}$$

7. A Simple Physical Interpretation of the Tangency and Detachment Conditions of a Stratified Vortex Layer

On the cone surface, the no-penetration condition holds:

$$v\cdot n_c=0 \tag{26}$$

Substituting the velocity components, we obtain:

$$u_r-u_z\,tan\,\alpha=0 \tag{27}$$

This condition ensures that the flow is tangent to the cone surface.

If we have $u_r-u_z\,tan\,\alpha>0$, (28)

the flow begins to deviate from the surface, corresponding to the onset of vortex-surface shedding. The model implies that:

$$\frac{dr}{d\theta}=a(z,t)=\mu\frac{U(z,t)\sin\alpha}{\Omega(z,t)}. \tag{29}$$

Thus, we arrive at the following conclusions:

- increasing α, leads to a stronger opening of the surface;
- increasing U, causes the surface to unwind more rapidly
- increasing Ω, results in tighter spiral turns;
- stratification, through **N,** affects the vertical coupling of the layers and the admissible forms of $U(z,t)$ and $\Omega(z,t)$.

8. Surface Derivatives

For surface (13), where $r(\theta,z,t)$ is taken from Eq. (12), and $r_c\,(z)$ from Eq. (8), we derive in the following analysis the normal vectors, the metric, the curvatures, and the condition for detachment from the cone.

Let us denote

$$r_\theta=\frac{\partial r}{\partial\theta}=a(z,t),$$
$$r_z=\frac{\partial r}{\partial z}=r_c'(z)+a_z(\theta-\theta_0)-a\,\theta_{0z}. \tag{30}$$

Since the tangent of the angle α is equal to $r_c'\,(z)$ (22), we have

$$\boxed{r_z=\tan\alpha+a_z(\theta-\theta_0)-a\,\theta_{0z}.} \tag{31}$$

From this we obtain the derivatives of the parametrization:

$$\mathbf{X}_\theta=\big(r_\theta\cos\theta-r\sin\theta,\ r_\theta\sin\theta+r\cos\theta,\ 0\big),$$
$$\mathbf{X}_z=\big(r_z\cos\theta,\ r_z\sin\theta,\ 1\big). \tag{32}$$

9. Unnormalized and unit normal vectors

Unnormalized normal vector:

$$\mathbf{N} = \mathbf{X}_\theta \times \mathbf{X}_z. \tag{33}$$

after calculation gives

$$\boxed{\mathbf{N} = \big(r_\theta \sin\theta + r\cos\theta,\ -(r_\theta\cos\theta - r\sin\theta),\ -r\,r_z\big)} \tag{34}$$

Or with $r_c = a$:

$$\boxed{\mathbf{N} = \big(a\sin\theta + r\cos\theta,\ -a\cos\theta + r\sin\theta,\ -r\,r_z\big).} \tag{34*}$$

Its length is given by:

$$|\mathbf{N}| = \sqrt{r^2 + r_\theta^2 + r^2 r_z^2}. \tag{35}$$

From this we obtain the unit normal vector:

$$\boxed{\mathbf{n} = \frac{1}{\sqrt{r^2 + r_\theta^2 + r^2 r_z^2}}\big(r_\theta\sin\theta + r\cos\theta,\ -r_\theta\cos\theta + r\sin\theta,\ -r\,r_z\big).} \tag{36}$$

10. First Fundamental Form of the Surface

The metric coefficients are:

$$E = \mathbf{X}_\theta \cdot \mathbf{X}_\theta = r_\theta^2 + r^2,$$

$$F = \mathbf{X}_\theta \cdot \mathbf{X}_z = r_\theta r_z, \tag{37}$$

$$G = \mathbf{X}_z \cdot \mathbf{X}_z = 1 + r_z^2.$$

From them we have:

$$\boxed{E = r^2 + a^2, \qquad F = a\,r_z, \qquad G = 1 + r_z^2.} \tag{38}$$

11. Second Fundamental Form

We need the second derivatives; thus, we have

$$\mathbf{X}_{\theta\theta} = \big((r_{\theta\theta} - r)\cos\theta - 2r_\theta\sin\theta,\ (r_{\theta\theta} - r)\sin\theta + 2r_\theta\cos\theta,\ 0\big),$$

$$\mathbf{X}_{\theta z} = \big(r_{\theta z}\cos\theta - r_z\sin\theta,\ r_{\theta z}\sin\theta + r_z\cos\theta,\ 0\big), \tag{39}$$

$$\mathbf{X}_{zz} = \big(r_{zz}\cos\theta,\ r_{zz}\sin\theta,\ 0\big).$$

For our model we have:

$$r_{\theta\theta} = 0, \qquad r_{\theta z} = a_z. \tag{40}$$

where are the coefficients of the second form:

$$e = \mathbf{n} \cdot \mathbf{X}_{\theta\theta}, \qquad f = \mathbf{n} \cdot \mathbf{X}_{\theta z}, \qquad g = \mathbf{n} \cdot \mathbf{X}_{zz}. \tag{41}$$

After simplification through the non-normalized normal **N**:

$$e^* = \mathbf{N} \cdot \mathbf{X}_{\theta\theta} = r(2a^2 + r^2),$$

$$f^* = \mathbf{N} \cdot \mathbf{X}_{\theta z} = r(a_z - r_z), \tag{42}$$

$$g^* = \mathbf{N} \cdot \mathbf{X}_{zz} = r\,r_{zz}.$$

Then we get for each e, f, g:

$$e = \frac{e^*}{|\mathbf{N}|}, \qquad f = \frac{f^*}{|\mathbf{N}|}, \qquad g = \frac{g^*}{|\mathbf{N}|}. \tag{43}$$

12. Gaussian and Mean Curvatures

The Gaussian curvature is defined by the formula:

$$K = \frac{eg - f^2}{EG - F^2}. \tag{44}$$

Since

$$EG - F^2 = (r^2 + a^2)(1 + r_z^2) - a^2 r_z^2 = r^2(1 + r_z^2) + a^2, \tag{45}$$

we obtain

$$\boxed{K = \frac{e^* g^* - (f^*)^2}{\big(r^2(1 + r_z^2) + a^2\big)^2}.} \tag{46}$$

That is, after substituting our expressions, we obtain an extended form for determining the Gaussian curvature at local points

$$\boxed{K = \frac{r^2(2a^2 + r^2)\,r_{zz} - r^2(a_z - r_z)^2}{\big(r^2(1 + r_z^2) + a^2\big)^2}.} \tag{47}$$

The mean curvature is given by:

$$H=\frac{eG-2fF+gE}{2(EG-F^2)}. \tag{48}$$

After substitution, we obtain a new working formula for the mean curvature

13. When the surface actually becomes developable

For a truly developable surface, the following condition must be satisfied:

$$\boxed{K=0.} \tag{50}$$

Thus, our spiral surface will be developable if

$$\boxed{(2a^2+r^2)\,r_{zz}-(a_z-r_z)^2=0.} \tag{51}$$

this condition is imposed on the functions$a(z,t)$ and $\theta_0\ (z,t)$.

In general, an arbitrary surface of the form (12) is not necessarily strictly developable; it provides a convenient kinematic model of a spiral sheet. For exact developability, an additional constrain $K\ =\ 0$ must be imposed.

14. Normal to the cone surface

The cone is defined by

$$F_c(r,z)=r-r_c(z)=0. \tag{52}$$

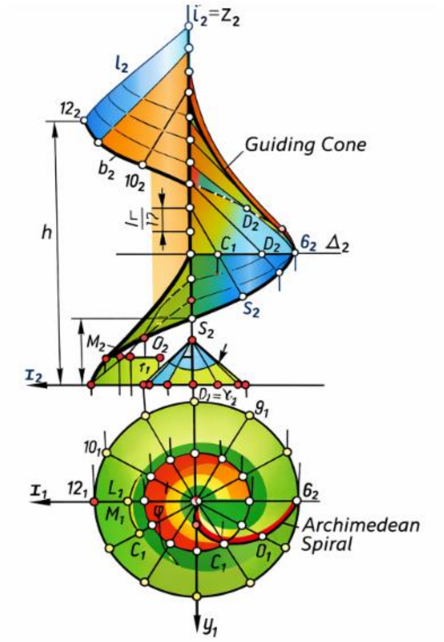

Fig. 1. This figure illustrates the geometric construction of a developable spiral surface obtained using a guiding cone.

In the cylindrical coordinate basis

$$\nabla F_c=\mathbf{e}_r-r_c'(z)\mathbf{e}_z=\mathbf{e}_r-\tan\alpha\,\mathbf{e}_z. \tag{53}$$

Unit normal to the cone surface:

$$\boxed{\mathbf{n}_c=\frac{\mathbf{e}_r-\tan\alpha\,\mathbf{e}_z}{\sqrt{1+\tan^2\alpha}}=(\cos\alpha)\,\mathbf{e}_r-(\sin\alpha)\,\mathbf{e}_z.} \tag{54}$$

In Cartesian coordinates $\boldsymbol{e_r}=(co\,s\,\theta\,,sin\,\theta,0)$, thus

$$\boxed{\mathbf{n}_c=\frac{(\cos\theta,\sin\theta,-\tan\alpha)}{\sqrt{1+\tan^2\alpha}}.} \tag{55}$$

15. Condition for separation of the lifting vortex sheet from a cone

We consider two conditions characterizing this state.

1) Geometric Tangency Condition

While the layer remains attached to the cone, its base line satisfies $r=\ r_c\ (z)$ (56)

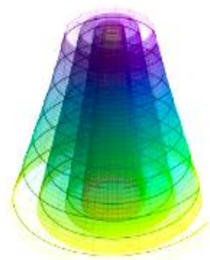

Fig. 2. A folded (streamlined) developable surface attached to the cone.

Thus, detachment occurs where the trajectory ceases to coincide with the cone surface.

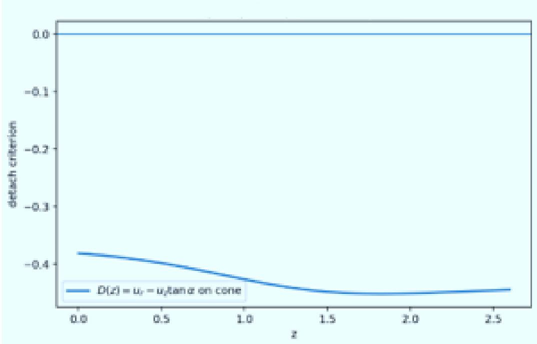

Fig. 3. The graph shows the dependence of the vortex-surface detachment criterion from the cone $D(z) = u_z - u_\theta \tan\alpha$, on the axial coordinate z.

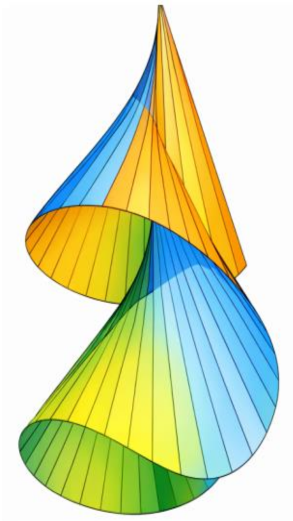

Fig. 4. The developable surface detached from the cone still retains the shape of the conical obstacle.

2) Kinematic No-Penetration Condition

On the cone, the following condition must hold:

$$\boxed{\mathbf{v} \cdot \mathbf{n}_c = 0.} \tag{57}$$

If we have

$$\boldsymbol{v} = u_r \boldsymbol{e_r} + u_\theta \boldsymbol{e_\theta} + u_z \boldsymbol{e_z} \tag{58}$$

since $\boldsymbol{e_\theta} \cdot \boldsymbol{n_c} = 0$,

$$\boxed{u_r - u_z \tan\alpha = 0} \tag{59}$$

on the cone surface. This expresses the **condition** that the flow **is tangent** to the cone (see: Fig.1 and 2.).

16. Condition for the actual detachment of the lifting vortex sheet

If the layer detaches from the cone, it may be assumed that the velocity along the surface is no longer able to maintain the layer in contact with the cone. A simple model is

$$\boxed{u_r - u_z \tan\alpha > 0} \tag{60}$$

where the flow deviates outward from the tangent to the cone. For an inviscid fluid, the shedding line is often defined through a critical value of the tangential shear:

$$\boxed{\left.\frac{\partial u_t}{\partial n}\right|_{cone} = 0} \tag{61}$$

as an analogue of the separation condition, where u_e denotes the tangential velocity along the cone and n is the normal to it.

For an inviscid vortex model, it is more convenient to employ a pressure-based criterion along the cone: separation begins where the pressure gradient ceases to maintain the attachment of the layer:

$$\boxed{\partial_s p = 0 \quad \text{or} \quad \partial_s p > 0} \tag{62}$$

depending on the chosen convention for the direction s.

17. Condition for the attachment of the surface to the cone along the shedding line

If $\mathbf{C}\,(s,t)$ is on the shedding line the following relation holds:

$$\boxed{\mathbf{C}(s,t) \in \Sigma_c,}$$
$$\boxed{\mathbf{n}(\mathbf{C}) \cdot \mathbf{n}_c(\mathbf{C}) = 0 \quad \text{or} \quad \mathbf{d}(\mathbf{C}) \in T\Sigma_c} \tag{63}$$

The formulation depends on whether the surface is specified through its normal vector or through its generating lines. In practice, geometric problems of surface dynamics of this type are more conveniently solved when the point of the shedding line lies on the cone, or when one of the tangent planes of the surface coincides with the tangent plane of the cone.

18. Conclusions

Thus, the proposed three-dimensional geometric model relates the shape of the developable spiral surface to several key parameters: the cone half-angle (α), the axial flow velocity $U(z,t)$, the angular velocity $\Omega(z,t)$, the shedding parameter μ, and the degree of fluid stratification.

The derived formulas make it possible to study the three-dimensional structure of unfolding vortex surfaces transporting matter that arise during the flow of swirling stratified fluids past cones, and can be used for further numerical modeling and stability analysis of such structures

Interestingly, a similar method of constructing a developable surface can be observed when peeling an orange from top to bottom along a spiral. The resulting 'cut and unfolded surface' of the peel retains the 'streamlined' shape of the orange.

This paper extends the possibilities for studying the conditions of formation of secondary semi-inertial open vortex domains (recirculation regions arising under contact interactions) that develop inside the main closed spheroidal vortex, which rotates with a certain angular velocity that is considered transverse in this case. The vortex possesses its own internal core, while the general concept for constructing such interacting counter-vortex cones is described in detail in papers [1–3]. Such classes of vortices differ significantly from the tubular vortex models proposed by Helmholtz in 1858 [9].

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