MATHEMATICAL SCIENCES

A SELF-SUSTAINED SPHEROIDAL-SPIRALOIDAL VORTEX WITH AN AXIAL FUNNEL

Milyute E.
Department of Mathematical Computer Science, Vilnius University and IRG"LITAVEM-3"
Milyus A.
IRG "LITAVEM-3"
Vilnius, Lithuania
https://doi.org/10.5281/zenodo.19590500

Abstract

This paper proposes a mathematical model of a new type of self-consistent vortex structure - a hybrid spheroidal-spiraloidal vortex with an internal spiral funnel. In contrast to classical models such as Hill's vortex and toroidal vortices [1-3], the proposed configuration represents a single continuous elliptical vortex region in which no distinct thin toroidal vortex tube is present. Instead, the internal flow is organized as a continuous axial channel - a funnel smoothly connected to the upper and lower parts of the vortex. It is shown that, within this framework, the source of motion, pressure, and density is transferred from the external flow to the internal structure: the axial flow and the associated angular momentum are responsible for the translational motion of the entire system. The velocity field is represented via a stream function decomposed into a Hill-type component and a funnel-induced component, and a Clebsch representation of the rotational potential is introduced, localized within the internal flow and the corresponding spiral vortex lines. Expressions are derived for the hydrodynamic impulse, circulation, streamline equations, angular momentum, and vortex moment. It is demonstrated that the existence of a self-consistent vortex is ensured by a closed cycle of fluid particles (micro-vortices), which acquire momentum in the funnel and subsequently move along single continuous three-dimensional spiral trajectories, encircling the internal region and returning to the base of the funnel without discontinuities or trajectory switching. This leads to the formation of a stable self-consistent structure in which flow closure, momentum transport, and pressure maintenance are governed by the internal geometry of motion, without the need to prescribe an external potential flow with continuous pressure [5-9].

Keywords: self-sustained vortex, spheroidal vortex, axial flow, internal funnel, spiral trajectories, vorticity, hydrodynamic impulse, circulation, Clebsch representation

1. Introduction. Let us consider an axisymmetric flow of an incompressible ideal fluid in cylindrical coordinates (r, φ, z), where the axis of symmetry coincides with the z-axis, and the center of the vortex structure moves along it such that $Z = \dot{Z} \cdot t$. The geometry of the flow domain is defined as a single spheroidal vortex region whose outer boundary has the form of a closed elliptical surface of Hill-type vortex. Within this region, the flow is organized not as a toroidal cavity or a thin vortex tube, but as a continuous internal structure with an axial channel - a funnel - smoothly connected to the upper and lower parts of the vortex. Thus, in a longitudinal cross-section, one observes not a thin-walled toroidal configuration, but a single connected vortex region with an internal axial flow. In this formulation, the external flow is not considered as the primary source of pressure, density, and driving impulse, since for vortex structures of this class, as noted already in Hill's classical formulation, the construction of an external flow with continuous pressure in closed form is difficult. Therefore, the source of motion, pressure redistribution, and density variation is transferred into the structure itself and is determined by the internal geometry of the flow. It is associated with the axial flow generated by the funnel and with a system of continuous spiral vortex lines filling the vortex volume. The forward motion of such a structure arises from the following mechanism. Fluid particles passing through the internal funnel acquire axial velocity and angular momentum, after which they transition onto smooth three-dimensional spiral trajectories within the vortex region. These trajectories continuously encircle the internal volume, pass through the lower region without discontinuities or switching between trajectory branches, then rise toward the upper part of the structure and return to the base of the funnel, forming a self-sustained cycle of motion. As a result, the ensemble of such trajectories forms a stable self-consistent flow, in which momentum transport and pressure maintenance are governed by the internal dynamics of the flow rather than by an externally imposed potential flow.

2. The velocity field is described by an axisymmetric stream function $\psi(r, z, t)$, for which the radial and axial velocity components are given by

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}\ u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}. \qquad (1)$$

The azimuthal component u_φ is allowed and plays an essential role, since it describes the rotational motion associated with the internal axial flow and the continuous spiral vortex lines filling the vortex volume. The total stream function is represented as

$$\psi = \psi_H + \psi_{fun}, \qquad (2)$$

where ψ_H describes the main closed elliptical region of Hill-type vortex, and ψ_{fun} represents the internal funnel embedded within this region and responsible for the axial flow. In contrast to models with a distinct toroidal structure, here ψ_H is defined as the stream function of a compact spheroidal vortex region, within which two main lower recirculation zones and two upper return zones are formed. It is convenient to introduce dimensionless variables $\xi = r/a$ and $\eta = (z - Z)/c$, where a and c define the characteristic transverse and longitudinal semi-axes of the elliptical region, and to define the spheroidal boundary by the condition

$$\sum_H : \xi^2 + \eta^2 = 1 \,. \tag{3}$$

The base part of the stream function can be chosen in the form

$$\psi_H = U_0 r^2 (1 - \xi^2 - \eta^2)[1 + \alpha_1 \eta + \alpha_2 \eta^2 + \alpha_3 \xi^2] \,, \tag{4}$$

where the coefficients α_i are selected such that the centers of the main recirculation cells are shifted downward relative to the geometric center, and the upper part of the flow provides return toward the funnel entrance. This formulation preserves the structure of Hill's vortex while allowing internal modification of the flow field to ensure matching with the axial channel. The funnel is defined by the internal surface $r = r_f(z - Z)$, where r_f is a smooth function that is small in the middle region and increases toward the upper and lower matching regions with the main core (kern) region. Its contribution to the stream function can be written as

$$\psi_{fun} = \frac{Q_f(t)}{2\pi} G\left(\frac{r}{r_f(z-Z)}, \frac{z-Z}{c}\right) \tag{5}$$

where $Q_f(t)$ is the flow rate through the funnel, and G is a smooth dimensionless function that vanishes on the funnel surface (i.e., equals zero at the funnel wall) and ensures continuous matching with the main Hill-type flow. As a result, the internal axial flow does not form a singular conical layer but rather a finite region of accelerated axial motion, consistent with the overall flow structure. In combination with the azimuthal velocity, this leads to the formation of continuous three-dimensional spiral trajectories of fluid particles, which pass through the axial region, encircle the internal volume, and return to the base of the funnel, thus forming a self-consistent flow structure within a single spheroidal vortex region.

3. The vorticity of the structure consists of two interrelated components. The first component is the azimuthal vorticity of the meridional flow

$$\omega_\varphi = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \tag{6}$$

which is related to the stream function by the equation

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = -r\omega_\varphi \tag{7}$$

This component governs the formation of the elliptical region of closed flow, within which the main recirculation zones are located. In accordance with the considered geometry, the vorticity ω_φ attains its maximum values in the lower part of the spheroidal region, where zones of enhanced vorticity corresponding to the recirculation regions are formed, and decreases toward both the outer boundary and the surface of the internal funnel. It is convenient to represent it in the form

$$\omega_\varphi = \omega_0 \frac{r}{a^2} g(\xi, \eta), \tag{8}$$

where g is a smooth function vanishing on the outer boundary and on the funnel surface, and attaining its maxima in regions of increased vorticity associated with the recirculation zones. The second component of vorticity is associated with the azimuthal swirl of the flow, distributed throughout the vortex volume and coupled with the internal axial flow. In contrast to models with a distinct toroidal structure, the vorticity here is not concentrated on discrete nested surfaces but is distributed as a continuous family of spiral vortex lines filling the region between the outer boundary and the axial funnel. These lines are naturally described by level surfaces $\lambda(r, z - Z) = \lambda_0$, where the function λ is chosen such that its level sets define nested flow layers consistent with the geometry of the spiral trajectories.

The azimuthal velocity can then be represented as

$$u_\varphi^{frame} = \Gamma(\lambda)/2\pi r \,, \tag{9}$$

where $\Gamma(\lambda)$ denotes the distribution of circulation across these layers. Within the internal funnel, a nonzero azimuthal velocity component is also present, ensuring the formation of a swirling axial flow and the transfer of angular momentum to the particles. At the same time, the axial component of vorticity remains nonzero, corresponding to the presence of an internal vortex core associated with the axial flow and the accompanying spiral motion. Under these conditions, the velocity field can naturally be represented in Clebsch form [1, 4]:

$$\mathbf{u} = \nabla\phi + \alpha\nabla\beta \,, \tag{10}$$

where ϕ describes the irrotational component of the internal mass transport, while the functions α and β define a continuous family of frozen-in vortex lines and the associated three-dimensional spiral structure of the flow. Thus, the rotational component of the velocity field is not given by a global scalar potential, but is determined by the internal geometry of the flow and the vorticity distribution associated with the axial funnel and the spiral trajectories filling the vortex volume.

4. The velocity potential, in the strict sense, is introduced only for the irrotational part of the motion. In the present formulation, the main flow is generated not by an external flow but by the internal structure of the flow. Therefore, the external potential ϕ_{ext} is allowed only as a weak auxiliary field satisfying Laplace's equation $\Delta\,\phi_{ext} = 0$ outside the vortex region and the condition of decay at infinity, but it is not considered as the primary carrier of pressure and does not determine the mechanism of vortex self-propulsion. The dominant contribution to the velocity field is given by the internal decomposition

$$\mathbf{u} = \nabla\phi_{int} + \mathbf{u}_{rot} \,, \tag{11}$$

where $\phi_{int} = \phi_H + \phi_{fun}$ describes the irrotational component of the flow within the spheroidal region, including the axial flow in the axial funnel, while $\mathbf{u}_{rot}$ corresponds to the rotational part associated with the internal swirl of the flow. The potential ϕ_{fun} describes the acceleration of particles in the axial funnel, their longitudinal transport, and subsequent redistribution in the upper part of the region. The potential ϕ_H represents the irrotational component of the recirculating flow within the main spheroidal region. The rotational part of the velocity field is naturally expressed in Clebsch form

$$\mathbf{u}_{rot} = \alpha\nabla\beta \,, \tag{12}$$

or, equivalently, through the decomposition of a rotational potential

$$\chi_{rot} = \chi_{fun} + \chi_{sp} \,, \tag{13}$$

where χ_{fun} is localized in the funnel region and is responsible for imparting angular momentum to the fluid particles, while χ_{sp} describes a spiral structure of vortex lines distributed throughout the volume, consistent with the geometry of the trajectories. Thus, the rotational component is not associated with a distinct localized structure but is distributed throughout the volume and determined by the internal geometry of the flow. It is this internal swirl that leads to pressure redistribution, sustaining the density distribution, and the formation of closed particle trajectories returning to the vortex core (kern). As a result, in this model the pressure is governed primarily by the internal distribution of velocity and vorticity, rather than by an externally imposed potential flow.

5. The hydrodynamic impulse of the considered vortex is directed along the axis of symmetry and is given by

$$\mathbf{P} = \frac{\rho}{2}\int_D \mathbf{r} \times \boldsymbol{\omega}\, dV\ . \qquad (14)$$

Due to axial symmetry, only the axial component is nonzero:

$$P_z = \pi\rho \int_D r^2 \omega_\varphi\, drdz. \qquad (15)$$

In contrast to classical models, where the impulse is associated with a distinct vortex structure, in the present formulation it is determined by the distribution of vorticity throughout the entire flow domain and is governed by the internal flow geometry. For analysis, it is convenient to decompose it into contributions from different flow regions:

$$P_z = P_z^H + P_z^{fun}\ , \qquad (16)$$

where P_z^H corresponds to the main spheroidal region with recirculating motion, and P_z^{fun} represents the contribution of the axial flow in the funnel. The dominant contribution to the impulse arises from the distribution of azimuthal vorticity within the volume:

$$P_z^H = \pi\rho \iint_{D_H} r^2 \omega_\varphi drdz, \qquad (17)$$

and reflects the contribution of the closed meridional circulation. The funnel contribution can be expressed either in terms of vorticity or, more transparently, in terms of axial mass transport:

$$P_z^{fun} = \rho \int_{D_{fun}} u_z dV. \qquad (18)$$

If the characteristic vorticity in the main region has the scaling $\omega_\varphi \sim \dot{Z}r/a^2$, then the corresponding contribution retains the Hill-type order of magnitude $\rho\dot{Z}a^4/c$, whereas the axial flow contribution scales as $\rho Q_f U_f$, where U_f is the mean velocity in the funnel. A key feature of the present model is that the directed hydrodynamic impulse is generated by internal redistribution of momentum. Fluid particles passing through the axial funnel acquire axial velocity and thereby contribute to a positive axial impulse. Subsequently, moving along continuous three-dimensional spiral trajectories, they encircle the internal region and return to the base of the funnel, thereby closing the flow without discontinuities. As a result, a stable self-consistent flow is established, in which the total impulse is maintained by internal dynamics rather than by external forcing. Thus, the translational motion of the vortex is governed not by external conditions, but by an internal mechanism of momentum transport associated with the axial flow and the corresponding spiral structure of particle trajectories.

6. Circulation in the present model also has a composite nature and is determined by the distribution of velocity and vorticity in different regions of the flow. The meridional circulation along a contour C_m, enclosing one of the recirculation zones is defined as

$$\boldsymbol{\Gamma}_m = \oint_{C_m} \mathbf{u} \cdot d\mathbf{l} = \iint_{\Sigma_m} \omega_\varphi\, dS\ . \qquad (19)$$

In the lower part of the spheroidal region, two such recirculation zones are formed, and in the symmetric case their corresponding circulations are equal in magnitude.

The azimuthal circulation is associated with the rotational component of the flow and is defined along circular paths of radius r:

$$\Gamma_\varphi = 2\pi r\, u_\varphi\ . \qquad (20)$$

It is not tied to a distinct geometric structure but characterizes the distributed swirl throughout the volume, associated with continuous spiral vortex lines. The circulation related to the axial funnel can be defined either along a contour enclosing the axial flow region or in terms of the flow rate and swirl within the funnel. In general, the total circulation can be expressed as

$$\Gamma = \Gamma_H + \Gamma_{fun}\ , \qquad (21)$$

where Γ_H is associated with the meridional recirculation in the main spheroidal region, and Γ_{fun} characterizes the contribution of the axial flow and the associated swirl. In contrast to classical models, the circulation here is not confined to an isolated region but is coupled with the internal transport of momentum through the axial channel. As a result, the circulation structure is determined not only by closed contours but also by the continuous redistribution of motion along spiral trajectories.

The streamlines are defined by the condition $\psi(r, z) = Const$. In a longitudinal cross-section, they form a single spheroidal region, in the lower part of which recirculation zones are located, in the upper part there are return-flow regions, and between them lies the axial funnel. On the funnel surface, the streamlines are tangent to its boundary; inside the funnel, they are stretched along the axis, then in the upper region they transition into diverging three-dimensional trajectories, encircle the internal volume, and return along lateral and lower paths to the base of the funnel. Thus, the entire flow is realized as a continuous system of closed spiral trajectories filling the vortex volume and ensuring a consistent redistribution of mass, momentum, and vorticity within a single coherent vortex structure.

7. Angular momentum. The angular momentum with respect to the z-axis is defined as

$$\boldsymbol{L}_z = \rho \int_D r\, u_\varphi\, dV\ . \qquad (22)$$

In the absence of azimuthal swirl $u_\varphi = 0$ and therefore $\boldsymbol{L}_z = 0$. However, in the present structure this condition is generally not satisfied, since the internal flow dynamics include a swirling axial motion and the associated three-dimensional spiral structure. In this formulation, the angular momentum is determined by the distribution of azimuthal velocity throughout the entire volume and can be represented as a sum of contributions associated with different flow regimes:

$$L_z = L_z^H + L_z^{fun}, \quad (23)$$

where L_z^H corresponds to the distributed swirl in the main spheroidal region, and L_z^{fun} represents the contribution of the axial flow in the funnel. In the main region, the angular momentum is given by

$$L_z^H = \rho \int_{D_H} r\, u_\varphi\, dV, \quad (24)$$

where the azimuthal velocity is related to the distribution of circulation across flow layers and reflects the continuous spiral structure of vortex lines. Within the internal funnel, for weak spiral swirl it is natural to assume

$$u_\varphi^{fun} = \Omega_f(r,z)\, r\,, \quad (25)$$

where Ω_f characterizes the local angular velocity associated with the axial flow. The corresponding contribution to the angular momentum is then

$$L_z^{fun} = \rho \int_{D_{fun}} \Omega_f r^2\, dV\,. \quad (26)$$

If the flow geometry defines a characteristic swirl angle β, then, in order of magnitude, $u_\varphi^{fun} \sim u_z \tan\beta$, and consequently $L_z^{fun} \sim \rho \tan\beta \int_{D_{fun}} r\, u_z\, dV$. Thus, the angular momentum is not a secondary effect but a key characteristic of the structure: it is through this quantity that the internal funnel transfers angular momentum to the particles, enabling their subsequent motion along closed spiral trajectories and their return to the base of the funnel. By analogy with a dipole moment, one may introduce the vortex moment:

$$\mathbf{M}_\omega = \frac{1}{2}\int_D \mathbf{r} \times \boldsymbol{\omega}\, dV\,, \quad (27)$$

which, for an axisymmetric configuration, is directed along the axis and is related to the hydrodynamic impulse by $\mathbf{M}_\omega \sim \mathbf{P}/\rho$ up to a normalization factor. The axial component of the vortex moment $\mathbf{M}_{\omega,z}$ can also be represented as a sum of contributions:

$$\mathbf{M}_{\omega,z} = \mathbf{M}_{\omega,z}^H + \mathbf{M}_{\omega,z}^{fun}\,, \quad (28)$$

where the first term characterizes the intensity of meridional circulation in the main region, and the second represents the contribution of the axial flow and the associated swirl. The stronger the internal swirl and the more efficiently the axial flow transfers angular momentum to the particles, the larger the vortex moment and the more stable the self-consistent translational motion of the structure. In this sense, the angular momentum and vortex moment serve as key integral characteristics reflecting the internal mechanism of vortex self-sustainment.

8. Mechanism of motion and geometry of particle trajectories.

The motion of particles in the considered hybrid spheroidal-spiraloidal vortex represents a smooth closed cycle consisting of three continuously connected stages: descent along the internal funnel, motion along the spheroidal surface, and return to the upper rim followed by repetition of the cycle. A key distinction from standard models is that passage through the lower region involves no discontinuities or switching between trajectories, but is realized as a single continuous curve. After descending through the funnel, a particle transitions onto the surface of the unit sphere

$$x^2 + y^2 + z^2 = 1, \quad (29)$$

where its motion is described as a smooth spherical spiral. This curve is parameterized by the polar angle θ, which varies from $\theta_{tip} = \arccos(z_{bot})$ through the value π, corresponding to the south pole, to $2\pi - \theta_{rim}$, where $\theta_{rim} = \arccos(Z_{top})$. The relations $r = |\sin\theta|$ и $z = |\cos\theta|$ ensure that the trajectory remains on the spherical surface.

This formulation guarantees continuous passage through the south pole: for $\theta > \pi$ the function $z = \cos\theta$ begins to increase, and the particle naturally transitions from descending to ascending motion without changing trajectory branch. This eliminates any possibility of trajectory switching, and the curve remains continuous and geometrically consistent.

The azimuthal angle φ is defined by an integral accumulation law and is a continuous function of θ, although its derivative behaves differently in the descending and ascending regions. For $\theta < \pi$ the angular velocity decreases according to an exponential law

$$d\varphi/d\theta = -\omega_{desc}(\theta), \quad (30)$$

where $\omega_{desc} = (2.4e^{-2u_{desc}} + 0.55)(1 + 0.08c)$, and

$$u_{desc} = (\theta - \theta_{tip})/(\pi - \theta_{tip}). \quad (31)$$

This law reflects the gradual decay of the swirl acquired in the funnel.

For $\theta > \boldsymbol{\pi}$, an inverse winding law applies

$$d\varphi/d\theta = -\omega_{asc}(\theta)\,, \omega_{asc} = n_0 R_{bot}^\alpha |\sin\theta|^\alpha\,, \quad (32)$$

which represents a geometric reflection of the funnel structure. If, within the funnel, the density of turns increases as the radius decreases, then on the sphere, for small values of $|\sin\theta|$ (near the pole), the turns are sparse, while they become denser as θ approaches θ_{rim}. This ensures a smooth transition of the trajectory toward the upper rim and consistency with the next cycle.

At $\theta = \pi$, the curve remains continuous and passes through the pole without a kink, since the variation of φ does not introduce coordinate discontinuities, and the parametrization preserves uniqueness of the trajectory. Thus, the entire second phase is a single smooth spherical spiral.

The total angular shift along the spherical segment is given by

$$\Delta\varphi_{geo} = -\int_{\theta_{tip}}^{\pi} \omega_{desc}\, d\theta - \int_{\pi}^{2\pi-\theta_{rim}} \omega_{asc}\, d\theta, \quad (33)$$

and, together with the contribution from the funnel descent, forms the total phase shift per cycle $\Delta\varphi_{cycle}$. If $\Delta\varphi_{cycle} \neq 2\pi k$, the trajectory does not close after a single revolution, and successive cycles form a quasiperiodic structure that gradually fills the vortex volume.

The mechanism of motion of such a structure is as follows. The internal funnel forms an axial acceleration channel in which particles acquire longitudinal velocity and angular momentum. After passing through the funnel, they transition onto continuous three-dimensional spiral trajectories that encircle the internal volume. These trajectories pass through the lower region without discontinuities, then rise to the upper part and return to the base of the funnel, closing the motion cycle.

Thus, each particle executes a continuous closed cycle, and the ensemble of such trajectories forms a stable internal flow. The translational motion of the entire structure arises from the net axial transport of momentum generated by the axial flow and redistributed throughout the volume via the spiral geometry of trajectories. In this formulation, the vortex constitutes a single self-consistent system in which the dynamics, pressure, and momentum transport are governed by the internal structure of the flow rather than by external potential forcing.

9. Conclusion

The developed model of a hybrid spheroidal-spiraloidal vortex demonstrates a fundamentally different mechanism of self-propulsion compared to classical vortex structures. In contrast to Hill's vortex, where motion is associated with an external potential flow, the present formulation shows that the dominant role is played by internal redistribution of momentum generated by the axial flow in the funnel and the associated three-dimensional swirl. Such a structure exists as a single self-consistent flow, in which the axial flow produces a directed impulse, while a continuous system of spiral vortex lines ensures the redistribution of momentum and closure of particle trajectories. The entire dynamics is governed by the internal geometry of the flow rather than by the presence of distinct structures or external flow conditions.

The obtained expressions for the hydrodynamic impulse, circulation, angular momentum, and vortex moment confirm that the translational motion of the vortex arises as a result of internal momentum transport along the axis followed by its redistribution throughout the volume.

Thus, the considered structure can be interpreted as a self-sustained vortex with an internal source of momentum, realized through the axial flow and continuous spiral particle trajectories. The proposed model may serve as a basis for further studies of the stability of such structures, their nonlinear dynamics under perturbations, and possible physical analogies in fluid dynamics, plasma, and other continuous media.

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