MATHEMATICAL SCIENCES

SOLUTION OF ONE MIXED PROBLEM FOR A PARABOLIC EQUATION WITH A DISCONTINUITY AT THREE POINTS

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Abstract

The presented work considers the process of heat transfer[10]. Thus, the coefficients of the equation have a discontinuty at the point $x = 0$ and this equation is used to find a solution to a mixed problem involving the heat transfer process[5]. The contour integral method was applied to solve theconsidering problem ([2],[3]).

Keywords: Bessel function,the contour integralmethod, mixed problem, spectral problem, eigenvalues, asymptotics

INTRODUCTION

A large number of mathematical problems of an applied nature are expressed and investigated with the help of the apparatus of partial differential equations. When constructing a mathematical description of real processes, models of second-order partial differential equations, characterized by discontinuous coefficients, often arise. The development of a mathematical model for heat distribution in environments with different heat transfer characteristics is also an extension of this approach and increases the significance of this type of problem from both practical and theoretical perspectives. The problem we are considering reflects this.

1.THE SOLUTİON OF ONE MİXED PROBLEM İN A CYLİNDRİCAL MEDİUM FOR AN EQUATİON WİTH DİSCONTİNUOUS COEFFİCİENTS.

The problem of finding a solution to the following equation is considered ([8],[10]):

$$\frac{\partial u^{(i)}}{\partial t} = \chi_i\left(\frac{\partial^2 u^{(i)}}{\partial x^2} + \frac{1}{x}\frac{\partial u^{(i)}}{\partial x}\right) + f^{(i)}(x,t), (i = \overline{1,3}) \tag{1}$$

which satisfies the following boundary condition:

$$u^{(1)}(0,t) = c_0$$
$$\left.\frac{\partial u^{(i)}}{\partial x}\right|_{x=a_i} + h_i\{u^{(i+1)}(a_i,t) - u^{(i)}(a_i,t)\} = 0, i = 1,2,$$
$$\left.\frac{\partial u^{(i)}}{\partial x}\right|_{x=a_i-1} + l_{i-1}\{u^{(i)}(a_{i-1},t) - u^{(i-1)}(a_{i-1},t)\} = 0, i = 2,3 \tag{2}$$
$$u^{(3)}(c,t) = 0,$$

also satisfy the initial condition

$$u^{(i)}(x,0) = \phi^{(i)}(x)\,, x \in (a_{i-1}, b_{i-1})\,. \tag{3}$$

Here constant numbers $\chi_i\,, h_i\ (i = \overline{1,3}), l_i\ (i = \overline{2,3})$ are parameters,the intervals $(a_{i-1}, b_{i-1})\ (i = 1,2,3)$ have common boundary points and do not intersect:

$$0 = a_0 < b_0 = a = a_1 < b_1 = b = a_2 < b_2 = a_3 = c.$$

Thus, the domain under consideration is divided into 3 parts $(0,a), (a,b)$ and (b,c), and these intervals converge only at the ends.

The function $f^{(i)}(x,t)$ in the equation is chosen in the following form:

$$f^{(i)}(x,t) == \begin{cases} 0, at\ x \in (0,a), \\ Ax, at\ x \in (a,b), \\ 0, at\ x \in (b,c). \end{cases} \tag{4}$$

and the initial functions $\phi^{(i)}(x)\ (i = \overline{1,3})$ are given as follows:

$$\phi^{(i)}(x)=\begin{cases}\alpha\, ln\dfrac{a}{b}+\beta, i=1, at\ x\in(0,a),\\ \alpha\, ln\dfrac{x}{b}+\beta, i=2, at\ x\in(a,b),\\ \gamma\, ln\dfrac{x}{c}+\delta, i=3, at\ x\in(b,c),\end{cases} \tag{5}$$

where $\alpha,\beta,\gamma,\delta$ are constants.

2.CONSTRUCTİNG OF THE GREEN'S FUNCTİON , CORRESPONDİNG TO SPECTRAL PROBLEM AND DETERMİNATİON OF THE EİGENVALUES

Consider the problem:

$$x_i\left(\frac{d^2y^{(i)}}{dx^2}+\frac{1}{x}\frac{dy^{(i)}}{dx}\right)-\lambda^2y^{(i)}=\begin{cases}0, i=1\\ \varphi^i(x), i=2,3\end{cases} \tag{6}$$

$$y^{(1)}(0,\lambda)=\frac{c_0}{\lambda}<\infty$$

$$\left.\frac{dy^{(1)}}{dx}\right|_{x=a}+h_1\left(y^{(2)}(a,\lambda)-y^{(1)}(a,\lambda)\right)=\frac{V_1}{\lambda^2}$$

$$\left.\frac{dy^{(2)}}{dx}\right|_{x=a}+l_1\left(y^{(2)}(a,\lambda)-y^{(1)}(a,\lambda)\right)=\frac{V_2}{\lambda^2}$$

$$\left.\frac{dy^{(2)}}{dx}\right|_{x=b}+h_2\left(y^{(3)}(b,\lambda)-y^{(2)}(b,\lambda)\right)=\frac{V_3}{\lambda^2}$$

$$\left.\frac{dy^{(3)}}{dx}\right|_{x=b}+l_2\left(y^{(3)}(b,\lambda)-y^{(2)}(b,\lambda)\right)=\frac{V_4}{\lambda^2}$$

$$\left.\frac{dy^{(3)}}{dx}\right|_{x=c}-h_3y^{(3)}(c,\lambda)=0 \tag{7}$$

This problem is called the spectral problem corresponding to the mixed problem.

In this section, we investigate the spectral problem depending on the parameter. The goal is to establish both a general solution and determine the corresponding spectral values. For this, it is appropriate to divide the problem into two separate auxiliary problems. As the first problem, let us find a solution to the homogeneous differential equation (6) that satisfies the given boundary conditions. Let us denote this solution as $y_1^{(i)}(x,\lambda)$. The second problem consists in finding a solution to equation (6) that satisfies the corresponding homogeneous boundary conditions. Let us denote this solution by $y_2^{(i)}(x,\lambda)$. In this case, it is obvious that the solution to the problem is the sum of the solutions həlli $y_1^{(i)}(x,\lambda)$ and $y_2^{(i)}(x,\lambda)$:

$$y^{(i)}(x,\lambda)=y_1^{(i)}(x,\lambda)+y_2^{(i)}(x,\lambda) \tag{8}$$

Solution of the first problem at $x\in(a_{i-1},b_{i-1})$ has the form:

$$y_1^{(i)}(x,\lambda)=\frac{\Delta_1^{(i)}(x,\lambda)}{\Delta(\lambda)}, (i=\overline{1,3}) \tag{9}$$

$$\frac{d^my_{11}^{(i)}}{dx^m}=\left(\frac{\lambda}{k_i}\right)^m\left\{\left(\frac{a_ik_i}{2x}\right)^{\frac{1}{2}}+\frac{\eta_{1m}^{(i)}(x)}{\lambda}+\frac{E_{1m}^{(i)}(x,\lambda)}{\lambda^2}\right\}e^{\frac{x-a_i}{k_i}\lambda}$$

$$\frac{d^my_{12}^{(i)}}{dx^n}=\left(-\frac{\lambda}{k_i}\right)^m\left\{\left(\frac{-a_ik_i}{2x}\right)^{\frac{1}{2}}+\frac{\eta_{1m}^{(i)}(x)}{\lambda}+\frac{E_{1m}^{(i)}(x,\lambda)}{\lambda^2}\right\}e^{\frac{-\lambda(x-a_i)}{k_i}} \tag{10}$$

Therefore, the analytically bounded functions funksiya $k_i=\sqrt{\chi_i}, i=2,3$, and $E_{jm}^{(i)}(x,\lambda)$ $(i=2,3;\ j=1,2;\ m=0,1,2)$ are functions with respect to the parameter λ in the expression, and these functions are also continuous on the intervals $x\in(a_{i-1},b_{i-1})$ $(i=2,3)$.

It is clear that the solution to the second problem is given by the Green's function ([2],[6]), where the solution is written as an integral expression:

$$y_2^{(i)}(x,\lambda)=\int_a^b G^{(i,2)}(x,\xi,\lambda)\varphi^{(2)}(\xi)d\xi++\int_b^c G^{(i,3)}(x,\xi,\lambda)\varphi^{(3)}(\xi)d\xi\ ,(i=\overline{1,3}) \tag{11}$$

where $G^{(i,j)}(x,\xi,\lambda)$ $(i=\overline{1,3}; j=\overline{2,3})$ is a Green function and

$$G^{(i,j)}(x,\xi,\lambda)=\frac{\Delta^{(i,j)}(x,\xi,\lambda)}{x_j\Delta(\lambda)}\ (i=\overline{1,3}; j=\overline{2,3}).$$

The asymptotics for $Re\lambda>0$ are defined in a similar way. Finally, the following asymptotic formula is obtained for the roots of the characteristic equation Δ(λ)=0 ([1],[4]).

$$\lambda_\nu=\frac{k_2k_3\nu\pi}{k_3(b-a)+k_2(c-b)}\left\{[1]+o\left(\frac{1}{\nu}\right)\right\}, \nu=0,\pm1,\pm2. \tag{12}$$

Using the obtained asymptotics, we can write the expression shown for the determinant $\Delta(\lambda)$ in the remaining part of the , λ-plane, except for the δ - neighborhood of the spectra given in ([7],[3]):

$$\left|\lambda^{4+\frac{1}{2}}\cdot e^{-\lambda\frac{a}{k_1}}\Delta(\lambda)\right|>N_\delta \qquad (13)$$

where N_δ is a constant that is positive and depends only on δ.

3. EVALUATION OF THE SOLUTION OF THE SPECTRAL PROBLEM

In order to find the functions $\Delta_1^{(i)}(x,\lambda)$ let us use the asymptotic expression of the Bessel function as well as the asymptotic expressions derived for the elements of the characteristic determinant [9]. As a result of this approach,we obtain in the cases where $i=1,2,3$:

$$\Delta_1^{(1)}(x,\lambda)=\lambda^{3+\frac{1}{2}}\left\{A_1^{(1)}(x,\lambda)e^{\lambda\frac{x-a}{k_1}}+A_2^{(1)}(x,\lambda)e^{-\lambda\frac{x-a}{k_1}}\right\}e^{-\lambda\left(\frac{a}{k_1}+\frac{b-a}{k_2}+\frac{c-b}{k_3}\right)}$$

$$\Delta_1^{(2)}(x,\lambda)=\lambda^{3+\frac{1}{2}}\left\{A_1^{(2)}(x,\lambda)e^{\lambda\frac{x-a}{k_2}}+A_2^{(2)}(x,\lambda)e^{-\lambda\frac{x-b}{k_2}}\right\} \qquad (14)$$

$$\Delta_1^{(3)}(x,\lambda)=\lambda^{3+\frac{1}{2}}\left\{A_1^{(3)}(x,\lambda)e^{\lambda\frac{x-b}{k_3}}+A_2^{(3)}(x,\lambda)e^{-\lambda\frac{x-c}{k_3}}\right\}e^{-\lambda\left(\frac{a}{k_1}+\frac{b-a}{k_2}+\frac{c-b}{k_3}\right)}$$

The first formula of these expressions is true in $x\in[\varepsilon,a)$, $\varepsilon>0$, the second formula is true in $x\in[a,b]]$ and the third is true in the interval $x\in[b,c]$. It can be concluded that for, $i=1$ at $x\in[\varepsilon,a]$; for $i=2,3$ at $x\in(a_{i-1};\ b_{i-1})$ (i-1); b_(i-1)) the following estimate is true for the solution of the first problem and its derivatives with respect to x up to the m-th order, under the condition $\lambda\in T_\nu$ [2]:

$$\left|\frac{1}{\lambda}\frac{d^m}{dx^m}y_1^{(i)}(x,\lambda)\right|\le\frac{M_1e^{\alpha_i\lambda}+M_2e^{\beta_i\lambda}}{|\lambda|^{2-m}} \qquad (15)$$

The parameters for $i=1,2,3;\ m=0,1,2$ are defined as follows:,

$$\alpha_1=\frac{x+a}{k_1};\ \beta_1=\frac{a-x}{k_1},x\in[\varepsilon;a]$$

$$\alpha_2=\frac{x-a}{k_2};\ \beta_2=\frac{b-x}{k_2},x\in[a;b]$$

$$\alpha_3=\frac{x-b}{k_3};\ \beta_3=\frac{c-x}{k_3},x\in[b;c]$$

The domain T_ν consists of the corresponding arcs of circles with radius r_ν, centered at the origin in the complex plane.

By applying the same method, the appropriate asymptotic behavior is determined for the solution of problem II. When $\operatorname{Re}\lambda<0$, the function $y_2^{(i)}(x,\lambda)$ and its derivatives have a similar type of exponential-decreasing and power-like evaluation.

$$\frac{d^my_2^{(1)}(x,\lambda)}{dx^m}=\lambda^{m-2}\left\{\left(\sum_{n=1}^{4}\mathcal{D}_{n_{11}}^{(m)}(x,\lambda)e^{\lambda\frac{x+a}{k_1}}+C_{n+1}^{(m)}(x,\lambda)e^{\lambda\frac{a-x}{k_1}}\right)\cdot\int_a^b[\sqrt{\xi}]\omega_{n_2}\varphi^{(2)}(\xi)d\xi\right.$$
$$\left.+\left(\sum_{n=1}^{3}\mathcal{D}_{n+2}^{(m)}(x,\lambda)e^{\lambda\frac{x+a}{k_1}}+C_{n+2}^{(m)}(x,\lambda)e^{\lambda\frac{a-x}{k_1}}\right)\cdot\int_a^b[\sqrt{\xi}]\omega_{n_3}\varphi^{(3)}(\xi)d\xi\right\},at\ x\in[\varepsilon,a]$$

$$\frac{d^my_2^{(2)}(x,\lambda)}{dx^m}=\lambda^{m-1}\left\{\left(\sum_{n=1}^{4}\mathcal{D}_{n_{21}}^{(m)}(x,\lambda)e^{\lambda\frac{x-c}{k_2}}++C_{n_{21}}^{(m)}(x,\lambda)e^{\lambda\frac{b-x}{k_1}}\right)\int_a^b[\sqrt{\xi}]\omega_{n_2}\varphi^{(2)}(\xi)d\xi+\right.$$
$$\left(\sum_{n=1}^{3}\mathcal{D}_{n_{22}}^{(m)}(x,\lambda)e^{\lambda\frac{x-c}{k_2}}++C_{n_{22}}^{(m)}(x,\lambda)e^{\lambda\frac{b-x}{k_2}}\right)\cdot\int_b^c[\sqrt{\xi}]\omega_{n_3}\varphi^{(3)}(\xi)-\frac{\varepsilon m_2}{x_2}\int_a^x\left(\frac{k_2}{2}\sqrt{\frac{\xi}{x}}+\right.$$
$$\left.\left.+\frac{1}{\lambda}\left[\eta_{m_1}^{(2)}(x,\xi)\right]\right)e^{\lambda\frac{x-\xi}{k_2}}\varphi^{(2)}(\xi)d\xi+(-1)^m\frac{\varepsilon m_2}{k_2}\int_x^b\left(-\frac{k_2}{2}\sqrt{\frac{\xi}{x}}++\frac{1}{\lambda}\left[\eta_{m_2}^{(2)}(x,\xi)\right]\right)e^{-\lambda\frac{x-\xi}{k_2}}\varphi^{(2)}(\xi)d\xi\right\},at\ x\in[a,b]$$

$$\frac{d^my_2^{(3)}(x,\lambda)}{dx^m}=\lambda^{m-1}\left\{\left(\sum_{n=1}^{4}\mathcal{D}_{n_{31}}^{(m)}(x,\lambda)e^{\lambda\frac{x-b}{k_3}}++C_{n_{31}}^{(m)}(x,\lambda)e^{\lambda\frac{c-x}{k_1}}\right)\int_a^b[\sqrt{\xi}]\omega_{n_2}\varphi^{(2)}(\xi)d\xi+\right.$$
$$\left(\sum_{n=1}^{3}\mathcal{D}_{n_{32}}^{(m)}(x,\lambda)e^{\lambda\frac{x-b}{k_3}}++C_{n_{32}}^{(m)}(x,\lambda)e^{\lambda\frac{c-x}{k_3}}\right)\cdot\int_b^c[\sqrt{\xi}]\omega_{n_3}\varphi^{(3)}(\xi)-\frac{\varepsilon m_3}{x_3}\int_b^x\left(\frac{k_3}{2}\sqrt{\frac{\xi}{x}}+\right.$$
$$\left.\left.+\frac{1}{\lambda}\left[\eta_{m_1}^{(3)}(x,\xi)\right]\right)e^{\lambda\frac{x-\xi}{k_3}}\varphi^{(3)}(\xi)d\xi+(-1)^m\frac{\varepsilon m_3}{k_3}\int_x^c\left(-\frac{k_3}{2}\sqrt{\frac{\xi}{x}}++\frac{1}{\lambda}\left[\eta_{m_2}^{(3)}(x,\xi)\right]\right)e^{-\lambda\frac{x-\xi}{k_3}}\varphi^{(3)}(\xi)d\xi\right\},x\in[b,c]$$

$$m=(0,1,2) \qquad (16)$$

Note that the functions , $\mathcal{D}_{mj}^{(m)}(x,\lambda)$ and $C_{mj}^{(m)}(x,\lambda)$ are bounded on the contours of $\mathcal{T}_\nu$ in the following cases: in the case of $i=1$ at $x\in[\varepsilon;a]$, and in addition, in the case of $i=2,3$ at $x\in[a_{i-1};b_{i-1}]$, where at $i=1,2,3$ and $j=1,2$ i=1,2,3 this is true.

As was emphasized, it is not immediately obvious that the solution of the spectral problem considered in the interval $x \in (0;\ a)$ is bounded on this boundary, since the coefficient of the equation is discontinuous at the point $x = 0$. Therefore, the solution in this interval was found and investigated only for values $x \in [\varepsilon; a]$ $(\varepsilon > 0)$.

In this section, we present an estimate of the function $y_1^{(1)}(x,\lambda)$ for the solution of the corresponding spectral problem in the interval $x \in (0, a)$:

$$\left|y_1^{(1)}(x,\lambda)\right| \leq \begin{cases} c_{15}|\lambda|^{-\frac{1}{2}}e^{-\left(\frac{x}{k_1}+\beta\right)Re\,\lambda}; \dfrac{\pi}{2} < \arg\lambda < \dfrac{3\pi}{2}, \\ c_{25}|\lambda|^{-\frac{1}{2}}e^{\left(\frac{x}{k_1}+\beta\right)Re\,\lambda};\ -\dfrac{\pi}{2} < \arg\lambda < \dfrac{\pi}{2}. \end{cases} \tag{17}$$

Then let's consider the second problem. Let's evaluate the solution $y_2^{(1)}(x,\lambda)$ of the given problem. In the entire domain of the complex λ--plane, we finally obtain this inequality for the solution corresponding to the case $i = 1$ of the problem:

$$\left|y_2^{(1)}(x,\lambda)\right| \leq \begin{cases} \dfrac{2BA}{|\lambda|^2}e^{-\left(\frac{x}{k_1}+\beta\right)Re\lambda}; \dfrac{\pi}{2} < \arg\lambda < \dfrac{3\pi}{2}, \\ \dfrac{2BA}{|\lambda|^2}e^{\left(\frac{x}{k_1}+\beta\right)Re\lambda};\ -\dfrac{\pi}{2} < \arg\lambda < \dfrac{\pi}{2}. \end{cases} \tag{18}$$

4. SOLUTION OF THE MIXED PROBLEM

In this section, an analytical representation of the solution of the given mixed-type problem in the form of a contour integral is given and it is justified that this expression satisfies both the boundary and initial conditions ([2],[3],[7]). Thus, for $x \in (a_{i-1}, b_{i-1})$ $(i = \overline{1,3})$ and $t > 0$, let us consider the equation

$$\frac{\partial p^{(i)}}{\partial t} = x_i\left(\frac{\partial^2 p^{(i)}}{\partial x^2} + \frac{1}{x}\frac{\partial p^{(i)}}{\partial x}\right). \tag{19}$$

In this problem, the boundary conditions for $t > 0$ are as follows:

$$\begin{gathered} p^{(1)}(0,t) = c_0 < \infty \\ \left.\frac{\partial p^{(1)}}{\partial x}\right|_{x=a} + h_1\left(p^{(2)}(a,t) - p^{(1)}(a,t)\right) = V_1 \\ \left.\frac{\partial p^{(2)}}{\partial x}\right|_{x=a} + l_1\left(p^{(2)}(a,t) - p^{(1)}(a,t)\right) = V_2 \\ \left.\frac{\partial p^{(2)}}{\partial x}\right|_{x=b} + h_2\left(p^{(3)}(b,t) - p^{(2)}(b,t)\right) = V_3 \\ \left.\frac{\partial p^{(3)}}{\partial x}\right|_{x=b} + l_2\left(p^{(3)}(b,t) - p^{(2)}(b,t)\right) = V_4 \\ \left.\frac{\partial p^{(3)}}{\partial x}\right|_{x=b} - h_3 p^{(3)}(c,t) = 0 \end{gathered} \tag{20}$$

On the other hand at $x \in (a_{i-1}, b_{i-1})$ initial condition has the form:

$p^{(1)}(x,0) = 0$, $p^{(2)}(x,0) = \varphi^2(x)$, $p^{(3)}(x,0) = \varphi^{(3)}$ (2.48)

Let's consider the following theorems:

Theorem 1. (Equation (19) has a unique solution that satisfies the boundary conditions and zero initial conditions, and this solution is given by the following formula:

$$p_1^{(i)}(x,t) = \frac{1}{\sqrt{\pi - i}}\int_S \frac{y_1^{(i)}(x,\lambda)}{\lambda}e^{\lambda^2 t}\,dt \tag{21}$$

where $y_1^{(i)}(x,\lambda)$ is the solution to the I spectral problem and $t > 0$, $x \in (a_{i-1}, b_{i-1})$, $i = 1,2,3$.

Theorem 2. Equation (19) has a unique solution satisfying homogeneous boundary conditions and initial conditions, and it has the following analytical expression:

$$p_2^{(i)}(x,t) = \frac{1}{\pi\sqrt{-1}}\int_S \lambda y_2^{(i)}(x,\lambda)e^{\lambda^2 t}\,d\lambda, \tag{22}$$

where $y_2^{(i)}(x,\lambda)$ is the solution to the II spectral problem, corresponding to (4.1)-(4.3).

Let S denote an infinitely extendable contour lying entirely in the domain $R_\delta = \left\{\lambda: |\lambda| > R\ |\arg\lambda| < \frac{\pi}{4} + \delta\right\}$ and whose infinitely distant parts coincide with the rays $\arg\lambda = \pm\left(\frac{\pi}{4} + \delta\right)$.

Consequently, the sum of these two solutions constitutes the general solution of the mixed problem. That is, the function

$$p^{(i)}(x,t) = p_1(x,t) + p_2(x,t) \tag{23}$$

is the required solution of the given mixed problem.

In this case, it is true that the function $p^{(i)}(x,t)$ also satisfies the first of the boundary conditions.

İn other words, for $i = 1$ the equality

$$p^{(1)}(0,t) = c_0 < \infty$$

must bu proven to be true. To do this, it is necessary to prove that the integrals mentioned below converge regularly for all $\lambda \in R_\delta$ in case of $x \in [0,a]$ and $t \in [0,T]$.

$$\int_S \frac{1}{\lambda} y_1^{(1)}(x,\lambda) e^{\lambda^2 t} d\lambda \text{ və } \int_S \lambda y_2^{(1)}(x,\lambda) e^{\lambda^2 t} d\lambda \tag{24}$$

Let us denote the intersection points of the contour S of the circles with radii r_n and $r_{n+\nu}$ and centered at the origin in the λ -complex plane as a_n, $a_{n+\nu}$; a_n' and $a_{n+\nu}'$, respectively.

In this case, to show that the given integrals

$$\int_{\overline{a_n a_{n+\nu}}} \frac{1}{\lambda} y_1^{(1)}(x,\lambda) e^{\lambda^2 t} d\lambda ; \quad \int_{\overline{a_n' a_{n+\nu}'}} \frac{1}{\lambda} y_1^{(1)}(x,\lambda) e^{\lambda^2 t} d\lambda$$

$$\int_{\overline{a_n a_{n+\nu}}} \lambda y_2^{(1)}(x,\lambda) e^{\lambda^2 t} d\lambda \quad \int_{\overline{a_n' a_{n+\nu}'}} \lambda y_2^{(1)}(x,\lambda) e^{\lambda^2 t} d\lambda \tag{25}$$

regularly converge it is sufficient to show that the integrals are sufficiently small when $|\lambda|$ takes sufficiently large values.

According to (16), we can write the following inequality:

$$\left| \int_{\overline{a_n a_{n+\nu}}} \frac{1}{\lambda} y_1^{(1)}(x,\lambda) e^{\lambda^2 t} d\lambda \right| \leq \int_{\overline{a_n a_{n+\nu}}} \frac{1}{|\lambda|} \left| y_1^{(1)}(x,\lambda) \right| \left| e^{\lambda^2 t} \right| |d\lambda| \leq\leq c_1 \int_{\overline{a_n a_{n+\nu}}} \frac{1}{|\lambda|^{\frac{3}{2}}} e^{\left(\frac{x}{k_1}+\beta\right) Re\lambda} e^{Re\lambda^2 t} |d\lambda|$$

$$= c_1 \int_{\overline{a_n a_{n+\nu}}} e^{\left(\frac{x}{k_1}+\beta\right) Re\lambda + Re\lambda^2 t} |d\lambda| \tag{26}$$

If $\lambda \in a_n a_{n+\nu}$, then

$$\left| e^{\lambda^2 t} \right| = e^{Re\lambda^2 t} = e^{|\lambda|^2 t \cos^2 \arg\lambda} = e^{|\lambda|^2 + \left(2\cos^2 \arg\lambda - 1\right)} = e^{-|\lambda|^2 t \left(1-2\delta^2\right)} .$$

Therefore, it is true that $\arg\lambda = \frac{\pi}{4} + \delta$ and $2\cos^2 \arg\lambda - 1 = -\varepsilon$ are true on this contour. We can then write:

$$\left| \int_{\overline{a_n a_{n+\nu}}} \frac{y_1^{(1)}}{\lambda} d\lambda \right| < c_1 \int_{\overline{a_n a_{n+\nu}}} \frac{\left(\frac{x}{k_1}+\beta\right) |\lambda| \cos\arg\lambda - |\lambda|^2 t}{|\lambda|^{\frac{3}{2}}} |d\lambda| =$$

$$= c_1 \int_{\overline{a_n a_{n+\nu}}} e^{\left(\frac{x}{k_1}+\beta\right)|\lambda|\delta - \left(1-2\delta^2\right)|\lambda|^2 t} \frac{d\lambda}{|\lambda|^{\frac{3}{2}}} \leq c_1 \int_{r_n}^{r_{n+\nu}} e^{-r\left(\left(1-2\delta^2\right)2t - \left(\frac{x}{k_1}+\beta\right)\delta\right)} dr <$$

$$c_1 \int_{r_n}^{\infty} e^{-r\left(\left(1-2\delta^2\right)rt - \left(\frac{x}{k_1}+\beta\right)\delta\right)} \frac{dr}{r^{\frac{3}{2}}} .$$

It is clear from this that if we take the expression $|\lambda| = r$ large enough for all $x \in [0,a]$, the following equality is satisfied:

$(1-2\delta^2) rt - \left(\frac{x}{k_1}+\beta\right)\delta > 0.$

From this we can say that the value of the integral

$$\int_{\overline{a_n a_{n+\nu}}} \frac{y_1^{(1)}(x,\lambda)}{\lambda} d\lambda$$

becomes sufficiently small for sufficiently large $|\lambda|$

The same analytical approach can be applied to other integrals. As a result, we can easily obtain the limit at $x \to 0$ in the first expression, and thus:

$$u_1^{(1)}(0,t) = \lim_{x\to 0} u_1^{(1)}(x,t) = \lim_{x\to 0} \frac{1}{\pi\sqrt{-1}} \int_S \frac{y_1^{(1)}(x,\lambda) e^{\lambda^2 t}}{\lambda} d\lambda = \frac{1}{\pi\sqrt{-1}} \int_S \frac{e^{\lambda^2 t}}{\lambda} \lim_{x\to 0} y_1^{(1)}(x,\lambda)\, d\lambda$$

$$= \frac{c_0}{\pi\sqrt{-1}} \int_S \frac{e^{\lambda^2 t}}{\lambda} d\lambda == \frac{c_0}{\pi\sqrt{-1}} \lim_{n\to\infty} \int_{S_n} \frac{e^{\lambda^2 t}}{\lambda} d\lambda = \frac{c_0}{2\pi\sqrt{-1}} \lim_{n\to\infty} \oint_{\Gamma_n} \frac{e^{\lambda^2 t}}{\lambda} d\lambda =$$

$$= \frac{c_0}{2\pi\sqrt{-1}} 2\pi\sqrt{-1} \left. \frac{e^{\lambda^2 t}}{1} \right|_{\lambda=0} = c_0 .$$

As a result, it is proven that the function $u_1^{(1)}(x,t)$), i.e. $u_1^{(1)}(0,t) = c_0$ completely satisfies the first boundary condition. By a similar argument, it can be shown that

$$\lim_{n\to\infty} \int_{\overline{a_n a_{n+\nu}}} \lambda y_2^{(1)}(x,\lambda) d\lambda = \lim_{n\to\infty} \int_{\overline{a_n' a_{n+\nu}'}} \lambda y_2^{(1)}(x,\lambda) d\lambda = 0 .$$

Therefore,

$$u_2^{(1)}(0,t) = \lim_{x\to 0} u_2^{(1)}(x,t) = \lim_{x\to 0} \frac{1}{\pi\sqrt{-1}} \int_S \lambda y_2^{(1)}(x,\lambda) e^{\lambda^2 t} d\lambda =$$

$$= \frac{1}{\pi\sqrt{-1}} \int_S \lambda e^{\lambda^2 t} y_2^{(1)}(x,\lambda) d\lambda = \frac{c_0}{\pi\sqrt{-1}} \int_S \lambda e^{\lambda^2 t} d\lambda = 0 .$$

The function involved in the integral is analytic and therefore its derivative is zero.

As a result, the function given in (23) satisfies all the boundary conditions in (20), including the initial condition at $x = 0$.

CONCLUSION

This paper examines a mixed problem for a parabolic differential equation with variable coefficients. The fundamental properties of the characteristic determinant of the corresponding spectral problem are analyzed, and the necessary estimates for the Green's function are substantiated. As a result, a solution to the mixed problem under consideration is found in the form of a contour integral in the class of piecewise continuous functions.

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