EXTENSION OF THE TOROIDAL TRANSPORT MODEL TO THE VON KÁRMÁN VORTEX STREET

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Abstract

In this paper, we formulate a reduced mathematical description of passive-scalar transport in a von Kármán vortex street by extending a local toroidal advection–filamentation model from an isolated coherent vortex to a spatially ordered chain of alternately signed vortices. The starting point is the observation that the vortex street is not itself a single invariant torus: the wake is time-dependent, vortices are shed, convected and deformed, and scalar material is repeatedly captured and released. Nevertheless, in a frame moving with an individual vortex, the scalar dynamics contains a near-integrable core with almost closed streamlines, surrounded by a strongly strained separatrix layer. This structure permits a multiscale description in which each coherent vortex carries a local action–angle dynamics, while weak non-integrable perturbations describe leakage, separatrix splitting and inter-vortex exchange. The resulting model combines three levels: local differential rotation and filamentation within a vortex, pairwise exchange between neighbouring vortices, and global transport along the wake. We derive the governing passive-scalar equations, introduce a Lamb–Oseen representation of the vortex street, formulate the local toroidal coordinates, obtain scaling laws for gradient growth and transverse filament thickness, and propose reduced amplitude equations for the evolution of scalar filaments. The theory is intended as a kinematic, analytically transparent analogue of a full Navier–Stokes simulation, and is particularly useful for interpreting Lagrangian computations of passive-scalar stretching in coherent vortex wakes.

Keywords: vortex street; passive scalar; chaotic advection; FTLE; Lagrangian coherent structures; filamentation; Hamiltonian dynamics; vortex tubes; mixing

1. Historical Background.

The study of the von Kármán vortex street dates back to the classical work of Theodore von Kármán [1], in which the regular pattern of alternating vortex shedding behind a bluff body was first established. It was shown that such a configuration represents a stable structure for flows at finite Reynolds numbers. In the following decades, the development of vortex theory, systematized in the monograph by Philip G. Saffman [2], provided a comprehensive framework for describing the dynamics of coherent vortex structures and their role in wake formation.

In parallel, it became clear that the transport of passive tracers in such flows is governed not only by turbulent effects, but also by the geometry of Lagrangian trajectories. The works of Hassan Aref [3] laid the foundation of the theory of chaotic advection, demonstrating that even deterministic laminar flows can produce exponential stretching of material lines. These ideas were further developed in the book by Julio M. Ottino [4], where a kinematic theory of mixing based on stretching and folding mechanisms was formulated.

With the development of Lagrangian methods for flow analysis, the concept of Lagrangian coherent structures (LCS) emerged as a central tool, formalized in the works of George Haller [5-6]. These studies showed that finite-time Lyapunov exponent (FTLE) fields reveal geometric structures that act as dynamical transport barriers. The theoretical framework and numerical methods for identifying such structures were further developed, in particular, by Shadden and co-authors [7].

At the same time, the theory of passive scalar transport relies on the classical results of George K. Batchelor [8], who showed that under strong shear conditions, scalar fields develop into thin filaments with concentration gradients amplified down to scales at which diffusion becomes dominant. This mechanism underlies the filamentation observed in vortex-dominated structures.

From a mathematical perspective, Lagrangian dynamics in two-dimensional incompressible flows is naturally formulated in Hamiltonian form, as rigorously established within the framework of classical mechanics by V.I.Arnold [9]. The development of the theory of perturbed integrable systems and the transition to chaos, presented in the works of Allan J. Lichtenberg and Michael A. Lieberman [10], provided a framework for describing the destruction of invariant tori and the emergence of chaotic advection in weakly non-integrable systems.

Against this background, the recent work of E. Milyute and A. Milyus [11] proposed a model of passive-scalar transport on toroidal stream surfaces, in which filamentation is driven by differential advection in an integrable vortex field. However, the application of this model to real flows, such as the von Kármán vortex street, requires accounting for unsteadiness and interactions between vortex structures.

The present work develops this idea and extends the toroidal model to the case of the von Kármán vortex street, interpreting it as a system of weakly perturbed local toroidal regions. Within each region, quasi-integrable dynamics is preserved, while interactions between vortices lead to the breakdown of invariant structures and the emergence of chaotic transport.

2. Introduction. The transport of a passive scalar in coherent vortical flows is one of the classical problems of fluid mechanics. A passive scalar may represent dye, temperature in the absence of buoyancy feedback, salinity in a kinematic approximation, pollutant concentration, smoke, plankton density, or any material marker whose evolution is dominated by advection. Even when the scalar does not affect the velocity field, its distribution can develop very fine spatial scales. These scales are produced not by molecular diffusion alone but by the geometry of the velocity field: stretching, folding, differential rotation, separatrix crossing and repeated capture by coherent structures.

The purpose of this article is to construct a mathematical model for such scalar transport in a von Kármán vortex street. The model is based on a toroidal picture of transport in which scalar material is first considered inside a single coherent vortex as if it were moving on a family of closed or nearly closed stream surfaces. In an ideal toroidal model, the scalar is stretched by differential angular advection: particles at different radii rotate with different angular velocities, so that an initially localized patch is transformed into a long, thin filament. In a real vortex street the situation is more complicated. There is no single stationary torus filling the wake. Instead, vortices of alternating sign are shed behind an obstacle, convected downstream, and subjected to the strain of the mean flow and of neighbouring vortices. The scalar is therefore not permanently confined to one invariant surface. It is captured by a vortex, wound into spirals, expelled through a deformed separatrix, and then possibly captured again by another vortex.

The central claim of this paper is that the toroidal model remains useful if it is interpreted locally and asymptotically. It is not a global model of the entire vortex street. Rather, it is a local model of transport inside one coherent vortex element, supplemented by weak perturbation terms representing interaction with the rest of the street. This interpretation leads naturally to a hierarchy of reduced equations. At the first level, an isolated vortex is approximated by an integrable action–angle system. At the second level, neighbouring vortices and the background wake introduce slow radial drift, separatrix deformation and chaotic advection. At the third level, the full vortex street becomes a chain of capture–stretch–escape events

The article is written in the style of a reduced hydrodynamic theory. The aim is not to solve the Navier–Stokes equations directly, but to derive a coherent transport model from a prescribed, time-dependent velocity field. This approach is analogous in spirit to long-wave and defect theories in shear flows: one keeps the essential geometric mechanisms while replacing the full system by equations that are easier to analyse. Here the essential mechanism is not the instability of the velocity field itself, but the stretching and redistribution of scalar material in a vortex street.

3. Governing passive-scalar problem. Let (x, y) denote the streamwise and cross-stream coordinates in the wake of a circular cylinder or a compact bluff body. The incompressible velocity field is written as

$$\mathbf{u}(x, y, z) = (u_x, u_y), \nabla \cdot \mathbf{u} = 0 \tag{1}$$

The passive scalar is denoted by $\chi(x, y, t)$. In the purely advective limit it satisfies:

$$(\partial\chi/\partial t) + \mathbf{u} \cdot \nabla\chi = 0 \tag{2}$$

If molecular diffusion is retained, the equation becomes:

$$(\partial\chi/t) + \mathbf{u} \cdot \nabla\chi = \kappa\Delta\chi\ , \tag{3}$$

where κ is the scalar diffusivity. In Lagrangian form, particle trajectories obey

$$dX/dt = u_x(X, Y, t), dY/dt = u_y(X, Y, t) \tag{4}$$

and the scalar value carried by a particle is conserved when $\kappa = 0$:

$$D\chi/Dt = 0\ . \tag{5}$$

The dimensionless control parameter measuring the relative importance of advection and diffusion is the Péclet number:

$$Pe = U_\infty D/\kappa\ , \tag{6}$$

where U_∞ is the incoming velocity and D is a characteristic obstacle diameter. In the large-Péclet regime, scalar gradients may become very large before diffusion becomes important. This is the regime in which filamentation is most visible.

Because the velocity field is incompressible, one may introduce a streamfunction ψ, such that

$$u_x = \partial\psi/\partial y\ , u_y = -\,\partial\psi/\partial x\ . \tag{7}$$

The passive scalar equation can then be written as:

$$\chi_t + \mathcal{J}(\psi, \chi) = \kappa\Delta\chi\ , \tag{8}$$

where $\mathcal{J}(f, g) = f_x g_y - f_y g_x$ is the Jacobian. This form emphasizes the Hamiltonian character of two-dimensional incompressible advection: in the absence of diffusion, particles move along trajectories generated by a time-dependent Hamiltonian $\psi(x, y, t)$.

4. Kinematic representation of the vortex street. A von Kármán street may be represented kinematically as a mean stream plus a sequence of localized vortices. We write:

$$\mathbf{u}(x, y, t) = U_\infty \mathbf{e}_x + \textstyle\sum_{n\in\mathbb{Z}} \boldsymbol{u}_n(x, y, t) + \mathbf{u}_b(x, y, t) \tag{9}$$

where $\mathbf{u}_b$ denotes the local correction due to the obstacle or near-wake region, and $\mathbf{u}_n$ is the velocity induced by the n-th vortex. A convenient smooth model is based on Lamb–Oseen vortices. The vorticity of the n-th vortex is approximated by

$$\omega_n(x, y, t) = (\Gamma_n/2\pi r_c^2)\exp[-\,h/2r_c^2] \tag{10}$$

where Γ_n is its circulation, r_c is the core radius, and $\big(X_n(t), Y_n(t)\big)$ is the vortex centre, $h = \big(x - X_n(t)\big)^2 + \big(y - Y_n(t)\big)^2$ (where **level sets** $h = const$ are *circles* centered at the vortex).

The total vorticity is then

$$\omega(x, y, t) \simeq \textstyle\sum_{n\in\mathbb{Z}} \Gamma_n G\big(x - X_n(t), y - Y_n(t)\big)\ , \tag{11}$$

where the kernel has a Gaussian form:

$$G(\xi, \eta) = (1/2\pi r_c^2)\exp[-\,(\xi^2 + \eta^2)/2r c_c^2]. \tag{12}$$

This representation allows the vortex street to be described as a superposition of smoothed localized vortices, which is convenient for the subsequent analysis of passive scalar transport. The induced tangential velocity of one Lamb–Oseen vortex is:

$$u_\theta(r) = (\Gamma_n/2\pi r)(1 - \exp[-\,r^2/2r_c^2]) \tag{13}$$

so that in Cartesian form

$$u_n(x, y, t) = u_\theta(r_n)\begin{pmatrix} -\,b/r_n \\ a/r_n \end{pmatrix}, \tag{14}$$

where $a = x - X_n(t), b = y - Y_n\,(t)$ and the distance to the vortex center is given by: $r_n = \sqrt{a^2 + b^2}$.

This expression describes the rotational velocity field around the vortex center, taking into account the smoothed (Gaussian) core.

The staggered street is represented by two rows of vortices,:

$$X_n^+(t) = nL + U_c t, Y_n^+ = m, \Gamma_n^+ = \Gamma_0 \; , (15)$$

and

$$X_n^-(t) = nL + (L/2) + U_c t, Y_n^- = -m, \Gamma_n^- = -\Gamma_0 (16)$$

Here L s the streamwise spacing between vortices of the same sign, $2m$ is the cross-stream spacing between the two rows, and U_c is the convection velocity of the vortex street. This representation is not a full solution of the Navier–Stokes equations, but it captures the geometrical features relevant for scalar transport: rotating cores, strain layers between vortices, and downstream drift.

A passive scalar initially localized near the wake may be specified as a Gaussian packet:

$$\chi(x, y, 0) = \exp[-\left((x - x_0)^2 + (y - y_0)^2\right)/2\sigma_0^2] \quad (17)$$

In a Lagrangian computation, the initial distribution is discretized into an ensemble of particles $\left(x_i(t), y_i(t)\right)$, each carrying a scalar weight χ_i. The Eulerian field is subsequently reconstructed using a smoothing kernel W:

$$\chi(x, y, t) \simeq \textstyle\sum_i \chi_i W\left(x - x_i(t), y - y_i(t)\right) (18)$$

This formula is also useful theoretically, because it makes clear that the apparent creation of scalar structure is a consequence of the deformation of the particle cloud.

5. Local co-moving coordinates near one vortex. Choose one vortex centre $\left(X_n(t), Y_n(t)\right)$. In the frame moving with that vortex we define:

$$\tilde{x} = x - X_n(t), \tilde{y} = y - Y_n(t) \; . \qquad (19)$$

The local polar coordinates are:

$$\rho = \sqrt{\tilde{x}^2 + \tilde{y}^2}, \phi = \tan^{-1}(\tilde{y}/\tilde{x}) \; . \qquad (20)$$

In these variables the Lagrangian equations can be written schematically as

$$\dot{\rho} = V_n(\rho, \phi, t), \dot{\phi} = \Omega_n(\rho, t) + W_n(\rho, \phi, t) \; , (21)$$

where Ω_n is the angular velocity generated by the vortex itself, while V_n and W_n contain non-axisymmetric effects, neighbouring vortices, mean shear, finite-core deformation and acceleration of the moving frame. For a Lamb–Oseen vortex,

$$\Omega_n(\rho) = u_\theta(\rho)/\rho = (\Gamma_n/2\pi\rho^2)(1 - \exp[-\rho^2/2r_c^2]). (22)$$

The ideal isolated-vortex limit is obtained by setting $V_n = 0, W_n = 0$. Then

$$\dot{\rho} = 0, \dot{\phi} = \Omega_n(\rho) \qquad (23)$$

and each circle $\rho = const$ is an invariant curve. This is the two-dimensional analogue of an invariant toroidal surface. The scalar is not mixed across ρ, but it is sheared along ϕ. If two particles have radii ρ and $\rho + \Delta\rho$, their angular separation after time t changes by:

$$\Delta\,\phi \; \simeq \; \Delta\phi(0) + \Omega_n'(\rho)\Delta\rho t. \qquad (24)$$

Thus an initially compact scalar patch is elongated into a spiral filament. This is the elementary toroidal mechanism: differential angular velocity converts radial thickness into azimuthal length. The passive scalar equation in local polar variables is:

$$\chi_t + V_n\chi_\rho + (\Omega_n + W_n)\chi_\phi = \kappa\left(\chi_{\rho\rho} + (1/\rho)\chi_\rho + (1/\rho^2)\chi_{\phi\phi}\right) . \qquad (25)$$

In the isolated inviscid limit this reduces to

$$\chi_t + \Omega_n(\rho)\chi_\phi = 0 \; , \qquad (26)$$

whose exact solution is:

$$\chi(\rho, \phi, t) = \chi_0(\rho, \phi - \Omega_n(\rho)t) \; . \qquad (27)$$

From this expression one directly obtains the growth of gradients. Differentiating with respect to ρ :

$$\chi_\rho(\rho, \phi, t) = \chi_{0\rho}(\rho, \phi - \Omega t) - t\Omega_n'(\rho)\chi_{0\phi}(\rho, \phi - \Omega t)(28)$$

so that for large time, provided $\Omega_n'(\rho) \neq 0$,

$$|\nabla\chi| \sim t|\Omega_n'(\rho)|\left|\chi_{0\phi}\right|. \qquad (29)$$

The transverse filament thickness therefore scales as

$$\ell_\perp(t) \sim \ell_0/(1 + |\Omega_n'(\rho)|t) \; , \qquad (30)$$

until diffusion arrests the cascade of scales. If diffusion is included, the smallest dynamically relevant thickness is obtained by balancing shear-induced thinning and diffusion. A standard estimate gives: $\ell_d \sim (\kappa/|\Omega_n'(\rho)|)^{1/3}$, which is the local Batchelor-type scale for shear-enhanced scalar gradients.

6. Toroidal variables and unwrapped representation. To make the analogy with toroidal transport explicit, introduce a radial action-like coordinate $\mathcal{J}$ and an angular coordinate θ. For an axisymmetric vortex a natural action variable is proportional to the area enclosed by a streamline:

$$\mathcal{J} = \; \rho^2/2 \; . \qquad (31)$$

Then

$$\dot{\mathcal{J}} = 0 \, , \dot{\theta} = \Omega(\mathcal{J}) \; . \qquad (32)$$

This is the canonical form of an integrable one-degree-of-freedom Hamiltonian system. The passive scalar equation becomes

$$\chi_t + \Omega(\mathcal{J})\chi_\theta = \kappa D_{\mathcal{J},\theta}\chi \qquad (33)$$

where $D_{\mathcal{J},\theta}$ is the Laplacian expressed in action–angle coordinates. For the inviscid problem:

$$\chi(\mathcal{J}, \theta, t) = \chi_0(\mathcal{J}, \theta - \Omega(\mathcal{J})t) \qquad (34)$$

In computations it is often useful to unwrap the local vortex into a rectangular domain. Define a periodic coordinate s, from the radial displacement relative to a reference ring ρ_0:

$$s = 1 + \alpha(\rho - \rho_0) \; (\mathrm{mod}\; 2) \; , \qquad (35)$$

and retain

$$\theta = \tan^{-1}(\tilde{y}/\tilde{x}). \qquad (36)$$

The scalar field then become $\chi(s, \theta, t)$. Spiral filaments in physical space appear as tilted bands in the (s, θ)-plane. The slope of these bands is controlled by the differential rotation:

$$d\theta/ds \sim (d\Omega/ds)t \; . \qquad (37)$$

Thus, the unwrapped picture is not merely a visualization; it is a diagnostic of the local shear. The increasing inclination of scalar bands in (s, θ) measures the accumulation of phase differences between neighbouring radial labels.

7. Perturbation by neighbouring vortices. The isolated toroidal model is integrable and therefore cannot describe inter-vortex transport. To model a vortex street, one must include weak non-axisymmetric perturbations. We write:

$$\dot{\mathcal{J}_n} = \varepsilon F_n(\mathcal{J}_n, \theta_n, t), \dot{\theta_n} = \Omega_n(\mathcal{J}_n) + \varepsilon G_n(\mathcal{J}_n, \theta_n, t) \; , (38)$$

where $0 < \varepsilon \ll 1$ measures the strength of the external strain imposed by neighbouring vortices and the mean wake. In a frame moving downstream with speed

U_c, one also introduces the streamwise coordinate s of the vortex street:

$$\dot{s} = U_c + \varepsilon H_n(\mathcal{J}_n, \theta_n, t) . \quad (39)$$

For $\varepsilon = 0$, the scalar remains confined to invariant curves $\mathcal{J}_n = const$. For $\varepsilon \neq 0$, the invariant curves are deformed. Near separatrices they may break, producing lobes through which scalar material is exchanged between vortex cores and the outer wake. The scalar equation in the perturbed action–angle variables is:

$$\partial\chi/\partial t + \varepsilon F_n(\partial\chi/\partial\mathcal{J}_n) + (\Omega_n(\mathcal{J}_n) + \varepsilon G_n)(\partial\chi/\partial\theta_n) + (U_c + \varepsilon H_n)(\partial\chi/\partial s) = \kappa\mathcal{D}\chi . \quad (40)$$

The small parameter ε has a clear physical meaning. If the characteristic velocity induced by the vortex core is $U_v \sim \Gamma_0/(2\pi r_c)$, and the characteristic strain due to a neighbouring vortex separated by distance d_v is $S \sim \Gamma_0/(2\pi d_v^2)$, then over a core radius r_c he perturbation velocity is of order Sr_c . Hence: $\varepsilon \sim Sr_c/U_v \sim (r_c/d_v)^2$. For compact vortices separated by more than a few core radii, ε is small inside the core but becomes order unity near the separatrix. This explains why the toroidal model is accurate in the vortex interior but must be supplemented by exchange dynamics near the boundary. A first-order expression for radial drift can be obtained by averaging over the fast angle. Suppose

$$F_n(\mathcal{J}, \theta, t) = \sum_{m,k} F_{mk}(\mathcal{J}) \exp\big(i(m\theta - k\omega_s t)\big) , \quad (41)$$

where ω_s is the vortex shedding frequency. Away from resonances, the average drift vanishes at first order:

$$\langle\dot{\mathcal{J}}\rangle = \varepsilon\langle F\rangle_\theta = 0 . \quad (42)$$

Near resonances satisfying

$$m\,\Omega(\mathcal{J}) \simeq k\omega_s , \quad (43)$$

slow secular exchange becomes possible. Thus scalar leakage is expected to be strongest near resonant shells and separatrix layers.

8. Reduced scalar-filament equation. Consider a scalar filament inside a vortex. Its centreline may be parameterized by the angular variable θ, while its local transverse thickness is $\ell(\theta, t)$. The simplest reduced description keeps only the stretching rate σ, the leakage rate λ, ands diffusion. Let $C(\theta, t)$ be the scalar amplitude along the filament. Then a one-dimensional model is

$$C_t + \Omega(\mathcal{J}_0)C_\theta = D_\ell C_{\theta\theta} - \lambda C , \quad (44)$$

where D_ℓ is an effective diffusivity along the filament and λ represents loss from the local vortex to the outer wake. The thickness evolves according to

$$d\ell/dt = -\sigma(t)\ell + \kappa/\ell , \quad (45)$$

where $\sigma(t) \sim |\Omega'(\mathcal{J}_0)|/(1 + |\Omega'(\mathcal{J}_0)|t)$ for pure shear winding. In the early inviscid stage,

$$\ell(t) \sim \ell_0/(1 + |\Omega'(\mathcal{J}_0)|t) . \quad (45^*)$$

In the diffusive stage, the balance $d\ell/dt \approx 0$ gives: $\ell(t) \sim (\kappa/\sigma(t))^{1/2}$. A more Eulerian reduced equation follows from expanding the scalar field in local sheared coordinates. Let

$$\Theta = \theta - \Omega(\mathcal{J})t . \quad (46)$$

Then

$$\partial_{\mathcal{J}}|_\theta = \partial_{\mathcal{J}}|_\Theta - t\Omega'(\mathcal{J})\,\partial_\Theta \quad (47)$$

Consequently, diffusion becomes anisotropically amplified:

$$\chi_t = \kappa\left[\partial_{\mathcal{J}}^2 - 2t\Omega'(\mathcal{J})\partial_{\mathcal{J}\Theta} + t^2\big(\Omega'(\mathcal{J})\big)^2\partial_\Theta^2 + \cdots\right]\chi . \quad (48)$$

The term proportional to t^2, is the mathematical expression of shear-enhanced dissipation. It shows that even small κ becomes effective after the scalar has been wound into sufficiently fine filaments.

9. Global capture–stretch–escape map. The vortex street consists of a sequence of coherent vortices. Instead of following the scalar continuously, one may construct a discrete map from one vortex encounter to the next. Let M_n denote the scalar distribution immediately after interaction with the n-th vortex. Then:

$$M_{n+1} = \mathcal{E}_{n+1}\mathcal{A}_n C_n M_n . \quad (49)$$

Here C_n is a capture operator, $\mathcal{A}_n$ is the intra-vortex advection and filamentation operator, and $\mathcal{E}_{n+1}$ is the escape and transfer operator. The advection operator is locally toroidal:

$$(\mathcal{A}_n\chi)(\mathcal{J}, \theta) = \chi(\mathcal{J}, \theta - \Omega_n(\mathcal{J})T_n) \quad (50)$$

where T_n is the residence time of scalar material in the vortex. The escape operator may be modelled by a probability kernel K_n:

$$(\mathcal{E}_n\chi)(\mathcal{J}', \theta') = \int K_n(\mathcal{J}', \theta' \mid \mathcal{J}, \theta)\,\chi(\mathcal{J}, \theta)d\mathcal{J}\,d\theta . \quad (51)$$

A simple leakage model is:

$$K_n(\mathcal{J}', \theta' \mid \mathcal{J}, \theta) = p_n(\mathcal{J}, \theta)\,\delta(\mathcal{J}' - \mathcal{J}_\varepsilon)\,\delta(\theta' - \theta_\varepsilon) + [1 - p_n(\mathcal{J}, \theta)]\,\delta(\mathcal{J}' - \mathcal{J})\,\delta(\theta' - \theta) \quad (52)$$

where p_n is the probability of escape during one turnover. The probability is small in the vortex core and large near the separatrix. A useful asymptotic form is:Type equation here.

$$p_n(\mathcal{J}, \theta) \sim \exp[-\Delta\mathcal{J}(\mathcal{J})/\varepsilon]\,P(\theta, t) , \quad (53)$$

where $\Delta\mathcal{J}(\mathcal{J})$ represents the barrier in the action variable, and $P(\theta, t)$ describes the dependence on angle and time, and where $\Delta\mathcal{J}$ measures the action distance from the separatrix. This expression captures the exponentially weak leakage from the core and the strong exchange near broken separatrices.

Thus, the global scalar field in the wake is therefore not merely a collection of isolated spirals. It is a concatenation of local winding events connected by escape kernels. The global stretching factor after N vortex encounters is approximately:

$$\Lambda_N \sim \prod_{n=1}^{N}(1 + |\Omega_n'(\mathcal{J}_n)|T_n)\,\Lambda_{\epsilon,n} , \quad (54)$$

where $\Lambda_{\epsilon,n}$ is the stretching associated with escape and inter-vortex strain. Taking logarithms gives the finite-time Lyapunov estimate:

$$\lambda_N \sim (1/\textstyle\sum_n T_n)\sum_{n=1}^{N}\log\big[(1 + |\Omega_n'(\mathcal{J}_n)|T_n)\,\Lambda_{\epsilon,n}\big] . \quad (55)$$

This formula separates local toroidal winding from global chaotic exchange.

10. Gradient growth and filament thickness. The numerical diagnostics associated with the model are the scalar gradient magnitude and the transverse filament thickness. The gradient magnitude is

$$|\nabla\chi| = \sqrt{\chi_x^2 + \chi_y^2} . \quad (56)$$

In a reconstructed particle field, the derivatives are computed after kernel smoothing. The maximum gradient is

$$G_{max}(t) = \max_{x,y}|\nabla\chi(x, y, t)|. \quad (57)$$

For pure differential rotation, the gradient grows algebraically:

$$G_{max}(t) \sim G_0(1 + |\Omega'|t). \quad (58)$$

In a vortex street, however, strain layers and repeated vortex encounters may produce an exponential envelope:

$$G_{max}(t) \sim G_0\,e^{\sigma t} \quad (59)$$

where σ is an effective stretching rate. The exponential law should not be interpreted as proof of fully developed turbulence; it is a finite-time consequence of repeated stretching and folding by coherent vortices. The transverse filament thickness may be estimated from a half-maximum width of scalar concentration in a local cross-section. If $\ell_\perp(t)$ denotes this width, then the toroidal model predicts

$$\ell_\perp(t) \sim \ell_0/(1+\alpha t) \quad (60)$$

during the shear-winding stage. In the vortex street, inter-vortex strain can accelerate thinning:

$$\ell_\perp(t) \sim \ell_0 e^{-\sigma t}\ , \quad (61)$$

until diffusion imposes the cutoff: $\ell_\perp \gtrsim \ell_d$. Thus, A combined empirical law is therefore:

$$\ell_\perp(t) \sim \max\{(\ell_0/(1+\alpha t))e^{-\sigma_\epsilon t}, \ell_d\} \quad (62)$$

where σ_ϵ measures additional stretching due to inter-vortex exchange.

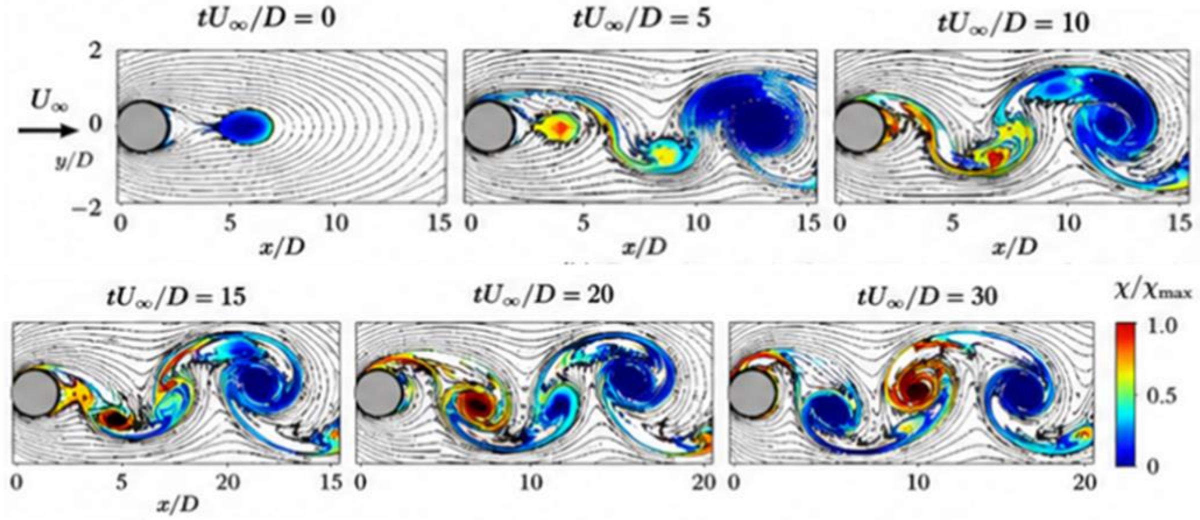

Fig. 1. Time evolution of a passive scalar in the wake behind a cylinder: vortex capture, spiral winding, and inter-vortex exchange in the von Kármán vortex street.

11. Relation to amplitude-equation thinking. The local toroidal transport model may be placed in a broader family of reduced hydrodynamic descriptions. In long-wave theories of shear flow, one often derives an amplitude equation for a slowly varying large-scale mode. In the present scalar-transport problem, the analogous amplitude is not a velocity instability amplitude but a coarse-grained scalar or filament-density amplitude.

Let $A(s,t)$ denote a slowly varying measure of scalar content along the wake. A minimal transport equation is:

$$A_t + U_c A_s = D_{eff} A_{ss} - \partial_s(V_{drift} A) - \Lambda\, A + Q\ . \quad (63)$$

Here D_{eff} is an effective longitudinal diffusivity generated by repeated vortex encounters, V_{drift} is a correction to mean convection, Λ is a net escape or dilution rate, and Q is a scalar source. This equation describes downstream redistribution but not the internal filament geometry.

To include filamentation, introduce a second field $B(s,t)$, representing mean scalar-gradient intensity, for instance:

$$B(s,t) = \langle |\nabla\, \chi|^2 \rangle_{cell}\ . \quad (64)$$

A phenomenological closure is:

$$B_t + U_c B_s = D_B B_{ss} + 2\sigma(s,t)B - 2\kappa C_B B^2 - \Lambda_B B\ . \quad (65)$$

The production term $2\sigma B$ represents stretching; the nonlinear sink $2\kappa C_B B^2$ represents diffusive destruction of gradients at small scales; and Λ_B represents dilution by escape from coherent structures. Such an equation is not universal, but it translates the geometric toroidal mechanism into an amplitude-equation framework.

A more direct analogue of reduced shear-flow theories is obtained near a separatrix. Let η be the coordinate normal to the separatrix and ξ the coordinate along it. The scalar equation in a thin exchange layer has the form:

$$\Theta_t + S\eta\Theta_\xi + V(\xi,t)\,\Theta_\eta = \kappa\Theta_{\eta\eta}\ , \quad (66)$$

where $S\eta$ is the local shear and V is a perturbation-induced normal velocity. This equation is structurally similar to boundary-layer or defect equations in which outer motion supplies a nonlocal forcing and inner dynamics regularizes sharp gradients. In the present context, the layer is not a vorticity defect but a scalar-exchange defect surrounding a coherent vortex.

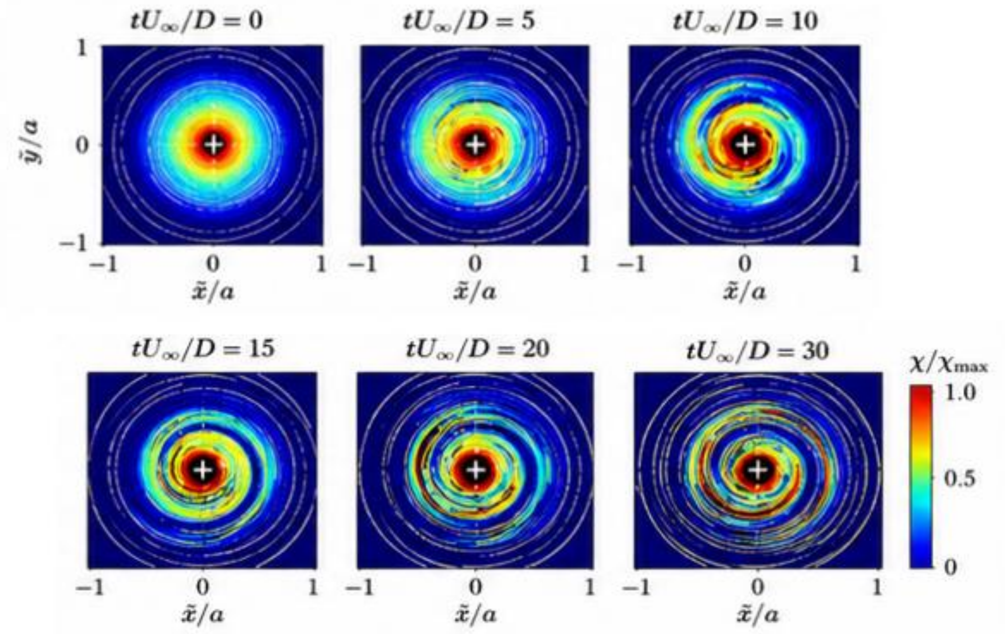

Fig. 2. Evolution of the passive scalar distribution in the co-moving vortex frame: winding of an initially compact patch into spiral filaments due to differential rotation.

12. Non-dimensional groups. The model contains several dimensionless parameters. The first is the Reynolds number of the wake:

$$Re = U_\infty D/\nu , \tag{67}$$

which controls vortex shedding and the persistence of coherent structures. The second is the Péclet number:

$$Pe = U_\infty D/\kappa , \tag{68}$$

which controls scalar filament survival.

The third is a vortex-strength parameter,

$$\Gamma^* = \Gamma_0/U_\infty D . \tag{69}$$

The fourth is a core-spacing parameter:

$$\delta = r_c/L , \tag{70}$$

and the fifth is a row-spacing parameter:

$$\beta = a/L . \tag{71}$$

The local shear rate inside a vortex is characterized by:

$$S_v = \max_{\rho} |\rho\, \Omega'(\rho)| . \tag{72}$$

A corresponding local scalar Péclet number is:

$$Pe_v = S_v r_c^2/\kappa . \tag{73}$$

When $Pe_v \gg 1$, the scalar forms thin spirals before diffusion smooths them. When $Pe_v = \mathcal{O}(1)$, diffusion suppresses fine filamentation and the toroidal mechanism is weak. The strength of inter-vortex perturbation is measured by:

$$\varepsilon \sim (r_c/d_v)^2 , \tag{74}$$

where d_v is the distance to the nearest neighbouring vortex. The asymptotic ordering underlying the local theory is: $Pe_v \gg 1$, $\varepsilon \ll 1$ inside the core, with $\varepsilon = \mathcal{O}(1)$ allowed in the separatrix layer.

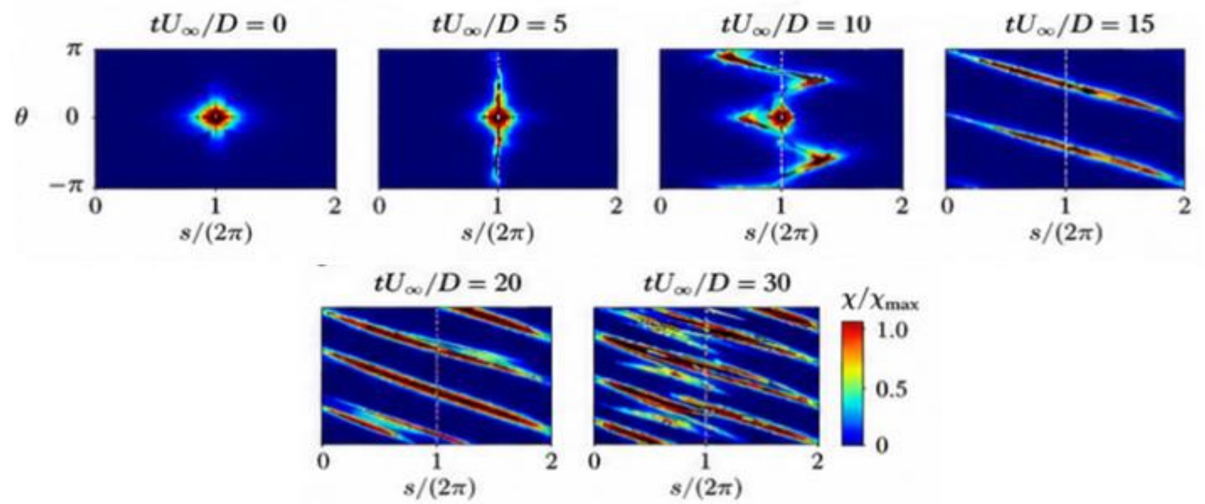

Fig. 3. Unwrapped representation of the scalar field in (s,θ) coordinates: differential rotation leads to the formation of inclined bands whose slope increases with time and reflects the local shear..

13. Mathematical statement of the model hierarchy. The complete reduced model can be summarized as follows. The full passive-scalar problem is:

$$\chi_t + \mathbf{u} \cdot \nabla \chi = \kappa \Delta \chi \quad (75)$$

with $\mathbf{u} = U_\infty \mathbf{e_x} + \sum_n \mathbf{u}_n + \mathbf{u}_b$ (76). Near the n-th vortex, introduce local action–angle variables $(\mathcal{J}_n, \theta_n)$. The local equations are:Type equation here.

$$\dot{\mathcal{J}}_n = \varepsilon F_n(\mathcal{J}_n, \theta_n, t)$$
$$\dot{\theta}_n = \Omega_n(\mathcal{J}_n) + \varepsilon G_n(\mathcal{J}_n, \theta_n, t) \quad (77)$$
$$\dot{s} = U_n + \varepsilon H_n(\mathcal{J}_n, \theta_n, t)$$

The scalar equation becomes:

$$\chi_t + \varepsilon F_n \chi_{\mathcal{J}_n} + (\Omega_n + \epsilon G_n)\chi_{\theta_n} + (U_c + \varepsilon H_n)\chi_s = \kappa \mathcal{D}_n \chi \ . \quad (78)$$

It should be noted that in the limit of the integrable core ($\varepsilon = 0$):

$$\chi_t + \Omega_n(\mathcal{J}_n)\chi_{\theta_n} = 0, \quad (79)$$

which corresponds to purely toroidal winding in the absence of inter-vortex exchange. Thus, here, in the integrable core limit we have:

$$\chi(\mathcal{J}_n, \theta_n, t) = \chi_0(\mathcal{J}_n, \theta_n - \Omega_n(\mathcal{J}_n)\ t) \ . \quad (80)$$

In the weakly perturbed case, scalar leakage is determined by resonances and separatrix splitting. In a global discrete representation:

$$M_{n+1} = \mathcal{E}_{n+1}\mathcal{A}_n C_n M_n \ . \quad (81)$$

The principal observables are: $G_{max}(t) = \max|\nabla\chi|$, $\ell_\perp(t) = \textit{local half-maximum filament width}$ and

$$\mathcal{M}(t) = \int |\nabla\chi|^2 dxdy \ , \quad (82)$$

where $\mathcal{M}$ is a scalar-gradient or mixing diagnostic. The expected asymptotic behaviour is

$$G_{max}(t) \sim G_0 e^{\sigma t} \quad (83)$$

for repeated vortex-street stretching, and $\ell_\perp(t) \sim \max\{\ell_0 e^{-\sigma t}, \ell_d\}$ (84), with algebraic thinning replacing exponential thinning inside a single isolated vortex.

14. Interpretation of numerical visualizations. The computational visualization associated with this model contains several panels, each corresponding to a level of the theory. The global scalar field $\chi(x, y, t)$ shows the passive scalar being advected downstream and distorted by the alternating vortices. At early times the scalar remains a compact cloud. At intermediate times it is captured by one or more vortex cores and wrapped into spiral-like structures. At later times the scalar is distributed through the wake, with thin filaments connecting neighbouring vortical regions.

A co-moving frame centered on a single vortex isolates the local toroidal transport mechanism. In this frame, the vortex core forms a nearly closed recirculation region, within which scalar contours wrap around the centre and progressively tighten due to differential rotation. This behaviour is described mathematically by:

$$\chi_t + \Omega(\rho)\chi_\phi = \kappa\Delta\chi \quad (85)$$

with weak perturbations accounting for inter-vortex interactions.

The unwrapped (s, θ) representation transforms the spiral structure into tilted bands. The progressive increase in the inclination of these bands provides direct evidence of shear-induced phase mixing. If the bands become broken, folded, or discontinuous, this indicates that the perturbation terms F and G are no longer negligible and that inter-vortex exchange has entered the local dynamics

The time series of $\max|\nabla\chi|$ quantifies the generation of small scales. Growth of this quantity indicates that scalar material is being stretched into increasingly sharp filaments. The time series of the transverse thickness $\ell_\perp$ measures the complementary contraction of these filaments. Together, these diagnostics test the predicted relation

$$G_{max}(t)\ell_\perp(t) \ \approx const \quad (86)$$

in the purely advective, locally area-preserving regime. Deviations from this relation indicate diffusion, leakage, or nonlocal stretching.

15. Discussion.

The toroidal extension proposed here should be interpreted as a geometric theory of transport rather than a complete model of the wake. It does not aim to predict the shedding frequency from first principles, nor does it provide a dynamically self-consistent Navier–Stokes solution for the vortex street. Instead, it assumes a prescribed kinematic vortex street and investigates how passive scalar material is transported within it. This separation is advantageous, as scalar transport is often governed more directly by the geometry of coherent structures than by the mechanisms responsible for their formation.

The model clarifies why a von Kármán vortex street mixes more effectively than a single steady vortex. A single steady vortex can stretch a scalar patch into a spiral, but in the absence of radial exchange the scalar remains confined on invariant curves. The resulting process is primarily filamentation rather than global mixing. In contrast, a vortex street introduces two essential mechanisms. First, neighbouring vortices generate strain that deforms invariant curves. Second, periodic shedding and downstream convection create repeated opportunities for capture and release. As a result, the scalar undergoes a sequence of local toroidal windings interspersed with separatrix-crossing events. This sequence significantly enlarges the accessible region in phase space compared to that of a single isolated vortex.

The model also explains why the strongest scalar gradients are expected near vortex boundaries and inter-vortex strain layers. Within the vortex core, the scalar is wound coherently. Near the separatrix, however, small perturbations can separate nearby trajectories and eject filaments into the wake. The interface between the coherent core and the surrounding strain field therefore becomes the natural location for the formation of sharp gradients.

From a mathematical perspective, the vortex street transforms an integrable action–angle system into a weakly non-integrable, time-periodic Hamiltonian system. In the unperturbed case, invariant curves $\mathcal{J} = const$ are preserved. In the perturbed case, resonances, separatrix splitting, and chaotic layers emerge. The passive scalar provides a direct visual representation of these structures: smooth spirals correspond to regular winding, while elongated inter-vortex filaments reflect chaotic exchange.

16. Conclusion.

We have developed an extension of a toroidal transport model to passive-scalar motion in a von Kármán vortex street. The main result is a hierarchy of reduced equations connecting three levels of description. At the local level, each coherent vortex contains nearly closed streamlines and supports toroidal-like differential rotation. In this region, an initially localized scalar patch is wound into a spiral, with gradient growth controlled by $\Omega'(\mathcal{J})$, and transverse thinning governed by $\ell_{\perp}(t) \sim [1 + |\Omega'|t]^{-1}$. At the intermediate level, neighbouring vortices perturb the local invariant curves and generate radial leakage, resonant exchange, and separatrix-layer stretching. At the global level, the wake acts as a sequence of capture, filamentation, escape and recapture events.

The mathematical model may be summarized by the perturbed action–angle system

$$\dot{\mathcal{J}}_n = \varepsilon F_n(\mathcal{J}_n, \theta_n, t)\ ,\ \dot{\theta}_n = \Omega_n(\mathcal{J}_n) + \varepsilon\ G_n(\mathcal{J}_n, \theta_n, t)\ ,$$

coupled to the passive-scalar equation:

$$\chi_t + \varepsilon F_n \chi_{\mathcal{J}_n} + (\Omega_n + \varepsilon G_n)\chi_{\theta_n} + (U_c + \varepsilon H_n)\chi_s = \kappa \mathcal{D}_n \chi$$

This formulation clarifies the sense in which the toroidal model applies to the vortex street. It does not represent the entire wake as a single torus; rather, each coherent vortex is treated as a local toroidal transport cell, while the remainder of the street acts as a weak but dynamically decisive perturbation. The resulting framework provides a compact mathematical interpretation of Lagrangian simulations and scalar-field visualizations in wakes: scalar patches are captured by vortices, wound into thin filaments, expelled through deformed separatrices, and transported downstream through a sequence of coherent structures.

A natural next step is to compare the reduced model quantitatively with particle-based simulations by measuring $\Omega(\mathcal{J})$, estimating ε, identifying separatrix layers, and fitting the observed evolution of $G_{max}(t)$ and $\ell_{\perp}(t)$. Such a comparison would convert the present geometric theory into a predictive reduced model of passive-scalar mixing in vortex streets.

References:

1. von Kármán, T. (1911). Über den Mechanismus des Widerstandes, den ein bewegter Körper in einer Flüssigkeit erfährt. Göttinger Nachrichten.
2. Saffman, P. G. (1992). Vortex Dynamics. Cambridge University Press.
3. Aref, H. (1984). Stirring by chaotic advection. J. of Fluid Mech., 143, 1–21.
4. Ottino, J. M. (1989). The Kinematics of Mixing: Stretching, Chaos, and Transport. Cambridge University Press.
5. Haller, G. (2001). Distinguished material surfaces and coherent structures in three-dimensional fluid flows. Physica D, 149, 248–277.
6. Haller, G. (2015). Lagrangian Coherent Structures. Annual Review of Fluid Mechanics, 47, 137–162.
7. Shadden, S. C., Lekien, F. & Marsden, J. E. (2005). Definition and properties of Lagrangian coherent structures from FTLE. Physica D, 212, 271–304.
8. Batchelor, G. K. (1959). Small-scale variation of convected quantities. J. of Fluid Mech., 5, 113–133.
9. Arnold, V. I. (1989). Mathematical Methods of Classical Mechanics. Springer
10. Lichtenberg, A. J. & Lieberman, M. A. (1992). Regular and Chaotic Dynamics. Springer.
11. Milyute E. & Milyus A. (2026) Advection and Filamentation of a Passive Scalar on Toroidal Stream Surfaces. In print.