SOLVING TRANSPORTATION PROBLEMS USING LINEAR PROGRAMMING IN MATHCAD

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Abstract

The paper provides basic theoretical information, example of solving the transportation problem using classical methods and the software environment MathCad. This development will contribute to more qualified training of specialists in economics and business.

Keywords: linear programming, MathCad, economic calculations.

Problem formulation. The transportation problem has an important place in linear programming and is widely used in the transportation and industry. It also can be used in some practical situations connected with resources management, creating of replacement schedule, appointment of employees, etc. Special importance it has for organization of rational supply of important cargoes as much as for optimal planning of cargo traffic and work of different types of transport. In recent years different facilities for engineering and scientific calculations have appeared. That enables specialists to solve posed problems without perfect knowing of programming languages, just using usual mathematical notation. However, it becomes necessary to master such software products as automated systems of engineering and economic calculations Excel and MathCad.

Review of previous studies. Some aspects of solving linear programming problems (including the transportation problem) in engineering calculations using MS Excel were described in [1, 2, 3]. However, methods of solving optimization and linear programming problems using modern computer technologies, in particular mathematical processor MathCad, are not developed enough. In Ukraine such scientists work over this problem: M.A. Martynenko, T.O. Kryvets, Ya.B. Petrivskii and others.

Purpose of the article is to propose methods of solving the linear programming transportation problem, as the most popular problem in economic calculations and a very important chapter in preparing of bachelors in the area of knowledge «Economy and business», using mathematical processor MathCad.

Main material description. In points $A_1, A_2 \ldots A_m$ contained homogeneous raw materials or goods that are needed to be transferred into points of consumption $B_1, B_2 \ldots B_n$. Reserves of supply points and needs of consumption points are known set values: $A=(a_1, a_2 \ldots a_m)$, $B = (b_1, b_2 \ldots b_n)$. Transportation costs from each supply point to each point of consumption characterized by a matrix:

$$\left(c_{ij}\right)=\begin{pmatrix} c_{11} & c_{12} & \cdot & \cdot & c_{1n} \\ c_{21} & c_{22} & \cdot & \cdot & c_{2n} \\ c_{31} & c_{32} & \cdot & \cdot & c_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{m1} & c_{m2} & \cdot & \cdot & c_{mn} \end{pmatrix}$$

Then, from the economic point of view the problem is posed as follows: transportation of raw materials or goods from supply points to consumption points should be planned so that demands of all the consumers were completely satisfied, all the reserves were taken out and, at the same time, total cost of all the transportations was the lowest possible.

To construct mathematical model of the problem we denote quantity of units of raw material or goods, that are planned to be transported from i - th supply point A_i to j-th point of consumption B_j, as x_{ij}. After that we obtain the following linear programming problem:

$$L=\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\cdot x_{ij} \to \min \tag{1}$$

$$\begin{aligned} &\sum_{i=1}^{m} x_{ij}=b_j \quad j=1...n \\ &\sum_{j=1}^{n} x_{ij}=a_i \quad i=1...n \end{aligned} \tag{2}$$

$$x_{ij}\geq 0 \quad i=1...n, \quad j=1...n \tag{3}$$

Definition 1. Plan of the transportation problem (1) – (3) is a set of values $x=x_{ij}$ $(i=1\ldots n, j=1\ldots n)$ that satisfies conditions (2) – (3).

Definition 2. Optimal plan of the transportation problem (1) – (3) is the plan $x^*=(x_{ij}^*)$ $(i=1\ldots n, j=1\ldots n)$, that satisfies condition (1).

Theorem 1. For the transportation problem to be solved it is necessary and sufficient to satisfy the balance condition

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

An algorithm of the method of potentials is based on fairness for the next theorem:

Theorem 2. If for some reference plan $X=(x_{ij})$, $(i=1\ldots n, j=1\ldots n)$ of the transportation problem exist such numbers α, β that

$$\beta_j - \alpha_i = c_{ij}$$

for $x_{ij} > 0$ and

$$\beta_j - \alpha_i \le c_{ij}$$

for $x_{ij} = 0$, then $X=(x_{ij})$ is the optimal plan of the transportation problem.

Definition 3. Numbers α_i, β_i are called potentials of supply and consumption points respectively.

Remark. If for the transportation problem (1) – (3) balance conditions are not satisfied, then we have an opened model of the transportation problem. In this case, we take an additional, fictitious supply (consumption) point with the quantity of reserves (needs) enough to satisfy the balance conditions. Transportation costs from this supply (consumption) point are equal to zero. After finding the solution, we discard the artificial components from the optimal plan.

To find the optimal plan of the transportation problem, we explore corresponding mathematical model using the method of potentials and built-in Mathcad functions.

Cost matrix Suppliers Consumers

$$\begin{bmatrix} 2 & 8 & 4 & 8 & 3 \\ 3 & 2 & 5 & 2 & 6 \\ 6 & 5 & 8 & 7 & 4 \\ 3 & 4 & 4 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 120 \\ 30 \\ 40 \\ 60 \end{bmatrix} \qquad \begin{bmatrix} 30 \\ 90 \\ 80 \\ 20 \\ 30 \end{bmatrix}$$

As $\sum_{i=1}^{4} a_i = \sum_{j=1}^{5} b_j = 250$, then the problem is balanced.

Using the north-west corner method we find an acceptable reference plan.

Table 1.

Suppliers	Consumers					Reserves
	B1	B2	B3	B4	B5	
A1	2 30	8 90	4	8	3	120 90 0
A2	3	2	5 30	2	6	30 0
A3	6	5	8 40	7	4	40 0
A4	3	4	4 10	2 20	1 30	60 50 30 0
Demand for resources	30 0	90 0	80 50 10 0	20 0	30	250 25

$$(1,1) \Rightarrow \min(30,120) = 30$$
$$(1,2) \Rightarrow \min(90,90) = 90$$
$$(2,3) \Rightarrow \min(80,30) = 30$$
$$(3,3) \Rightarrow \min(50,40) = 40$$
$$(4,3) \Rightarrow \min(10,60) = 10$$
$$(4,4) \Rightarrow \min(20,50) = 20$$
$$(4,5) \Rightarrow \min(30,90) = 30$$

The reference plan is obtained. To this plan correspond next transportation costs:

$$F = 30 \cdot 2 + 90 \cdot 8 + 30 \cdot 5 + 40 \cdot 8 + 10 \cdot 4 + 20 \cdot 2 + 30 \cdot 1 = 1360$$

The obtained reference plan is not the optimal. For its optimization we use the method of potentials. To determine potentials of suppliers and consumers we construct a system of equations for filling cells of the table 2.

$$c_{ij} = u_i + v_j$$

$$\begin{cases} u_1 + v_1 = 2 \\ u_1 + v_2 = 8 \\ u_2 + v_3 = 5 \\ u_3 + v_3 = 8 \\ u_4 + v_3 = 4 \\ u_4 + v_4 = 2 \\ u_4 + v_5 = 1 \end{cases} \quad \begin{array}{ll} u_1 = 0 & v_1 = 2 - 0 = 2 \\ u_2 = 5 - 4 = 1 & v_2 = 8 - 0 = 8 \\ u_3 = 8 - 4 = 4 & v_3 = 4 - 0 = 4 \\ u_4 = 0 & v_4 = 2 - 0 = 2 \\ & v_5 = 1 - 0 = 1 \end{array}$$

This undefined system has 7 equations and 9 unknowns therefore to solve it we give an arbitrary value to one of the potentials. Values of potentials we write into the table 2.

Table 2.

Suppliers	Consumers					Reserves	u_i
	B1	B2	B3	B4	B5		
A1	2 30	8 **90 160**	4 **0 30**	8 **-6**	3 **-2**	0	0
A2	3 **0**	2 **7 30**	5 **30 10**	2 **1**	6 **-4**	0	1
A3	6 **0**	5 **7**	8 40	7 **-1**	4 **1**	0	4
A4	3 **-1**	4 **4**	4 10	2 20	1 30	0	0
Needs	0	0	0	0	0		
v_j	2	8	4	2	1		

Next, we evaluate free cells $\Delta_i = u_i + v_j - c_{ij}$.

$$\Delta_{13} = 0 + 4 - 4 = 0 \qquad \Delta_{21} = 1 + 2 - 3 = 0$$
$$\Delta_{14} = 0 + 2 - 8 = -6 \qquad \Delta_{22} = 1 - 8 - 2 = 7$$
$$\Delta_{15} = 0 + 1 - 3 = -2 \qquad \Delta_{24} = 1 + 2 - 2 = 1$$
$$\Delta_{25} = 1 + 1 - 6 = -4$$
$$\Delta_{31} = 4 + 2 - 6 = 0$$
$$\Delta_{32} = 4 + 8 - 5 = 7 \qquad \Delta_{41} = 0 + 2 - 3 = -1$$
$$\Delta_{34} = 4 + 2 - 7 = -1 \qquad \Delta_{42} = 0 + 8 - 4 = 4$$
$$\Delta_{35} = 4 + 1 - 4 = 1$$

$\max(\Delta > 0) = ?$ – we choose (2,2), $\Delta_{22} = 7 > 0$, $\min(90{,}30) = 30$

Values in cells have changed.

Table 3.

	B1	B2	B3	B4	B5
A1	2 30	8 **60**	4 **30**	8	3
A2	3	2 **30**	5 **0**	2	6
A3	6	5	8 40	7	4
A4	3	4	4 10	2 20	1 30

$$\begin{cases} u_1+v_1=2 \\ u_1+v_2=8 \\ u_1+v_3=4 \\ u_2+v_2=2 \\ u_3+v_3=8 \\ u_4+v_3=4 \\ u_4+v_4=2 \\ u_4+v_5=1 \end{cases} \quad \begin{array}{l} u_1=0 \quad v_1=2 \\ u_2=2-8=-6 \quad v_2=8 \\ u_3=8-4=4 \quad v_3=4 \\ u_4=0 \quad v_4=2-0=2 \\ \qquad v_5=1-0=1 \end{array}$$

$$\Delta_{21}=-6+2-3=-7$$

$$\Delta_{14}=0+2-8=-6 \quad \Delta_{23}=-6+4-5=-7$$

$$\Delta_{15}=0+1-3=-2 \quad \Delta_{24}=-6+2-2=-6$$

$$\Delta_{25}=-6+1-6=-11$$

$$\Delta_{31}=4+2-6=0$$

$$\Delta_{32}=4+8-5=7>0 \quad \Delta_{41}=0+2-3=-1$$

$$\Delta_{34}=4+2-7=-1 \quad \Delta_{42}=0+8-4=4>0$$

$$\Delta_{35}=4+1-3=2>0$$

We skip 2 iterations. After the 4th iteration we obtain the next plan:

$$F=30\cdot 2+80\cdot 4+10\cdot 3+30\cdot 2+40\cdot 5+20\cdot 4+20\cdot 2+20=810.$$

Let’s find the solution of the transport problem in the software environment Mathcad.

ORIGIN := 1

$$X(x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x12,x13,x14,x15,x16,x17,x18,x19,x20) := \begin{pmatrix} x1 & x2 & x3 & x4 & x5 \\ x6 & x7 & x8 & x9 & x10 \\ x11 & x12 & x13 & x14 & x15 \\ x16 & x17 & x18 & x19 & x20 \end{pmatrix}$$

$$C := \begin{pmatrix} 2 & 8 & 4 & 8 & 3 \\ 3 & 2 & 5 & 2 & 6 \\ 6 & 5 & 8 & 7 & 4 \\ 3 & 4 & 4 & 2 & 1 \end{pmatrix} \qquad A := \begin{pmatrix} 120 \\ 30 \\ 40 \\ 60 \end{pmatrix} \qquad B := \begin{pmatrix} 30 \\ 90 \\ 80 \\ 20 \\ 30 \end{pmatrix}$$

$$\begin{aligned} F(x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x12,x13,x14,x15,x16,x17,x18,x19,x20) := {} & 2\cdot x1+8\cdot x2+4\cdot x3 \ldots \\ & + \blacksquare + 8\cdot x4+3\cdot x5+3\cdot x6+2\cdot x7+5\cdot x8+2\cdot x9+6\cdot x10 \ldots \\ & + \blacksquare + 6\cdot x11+5\cdot x12+8\cdot x13+7\cdot x14+4\cdot x15+3\cdot x16 \ldots \\ & + \blacksquare + 4\cdot x17+4\cdot x18+2\cdot x19+1\cdot x20 \end{aligned}$$

x1 := 1 x2 := 1 x3 := 1 x4 := 1 x5 := 1 x6 := 1 x7 := 1 x8 := 1 x9 := 1

x10 := 1 x11 := 1 x12 := 1 x13 := 1

x14 := 1 x15 := 1 x16 := 1 x17 := 1 x18 := 1 x19 := 1 x20 := 1

Given

$$x1 + x2 + x3 + x4 + x5 = 120$$

$$x6 + x7 + x8 + x9 + x10 = 30$$

$$x11 + x12 + x13 + x14 + x15 = 40$$

$$x16 + x17 + x18 + x19 + x20 = 60$$

$$x1 + x6 + x11 + x16 = 30$$

$$x2 + x7 + x12 + x17 = 90$$

$$x3 + x8 + x13 + x18 = 80$$

$$x4 + x9 + x14 + x19 = 20$$

$$x5 + x10 + x15 + x20 = 30$$

$x1 \geq 0$ $x2 \geq 0$ $x3 \geq 0$ $x4 \geq 0$ $x5 \geq 0$ $x6 \geq 0$ $x7 \geq 0$ $x8 \geq 0$ $x9 \geq 0$ $x10 \geq 0$

$x11 \geq 0$ $x12 \geq 0$ $x13 \geq 0$ $x14 \geq 0$ $x15 \geq 0$ $x16 \geq 0$ $x17 \geq 0$ $x18 \geq 0$ $x19 \geq 0$ $x20 \geq 0$

$$\begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \\ x5 \\ x6 \\ x7 \\ x8 \\ x9 \\ x10 \\ x11 \\ x12 \\ x13 \\ x14 \\ x15 \\ x16 \\ x17 \\ x18 \\ x19 \\ x20 \end{pmatrix} := \text{Minimize}(F, x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, x11, x12, x13, x14, x15, x16, x17, x18, x19, x20)$$

$$F(x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, x11, x12, x13, x14, x15, x16, x17, x18, x19, x20) = 810$$

Conclusions. In this paper, detailed solution for the transportation problem, that uses the automated system of engineering and economic calculations Mathcad, is given. The authors hope that introduced developments will contribute to training of highly qualified specialists in economics, marketing, management, accounting and auditing, especially in conditions of limited classroom hours for studying informatics.

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