MATHEMATICAL SCIENCES

THE MICROVORTEX CIRCULATION VECTOR AS AN INTEGRAL CHARACTERISTIC OF COHERENT VORTEX FILAMENTS

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https://doi.org/10.5281/zenodo.21982207

Abstract

A geometric and hydrodynamic model of a unified ellipsoidal vortex with a central concave helical kern is developed from the viewpoint of the integral rotational characteristics of vortex filaments. The principal object introduced in this work is the microvortex circulation vector, $\boldsymbol{\mu} = \Gamma\,\mathbf{T}$, which combines the local circulation with the tangent orientation of an individual material filament. Unlike the classical vorticity field, the proposed vector represents the oriented flux of vorticity through the transverse cross-section of a vortex tube and therefore provides an integral description of the local rotational state. The evolution of the microvortex circulation vector is derived for both ideal and viscous fluids. It is shown that, under Kelvin circulation conservation, the magnitude of $\boldsymbol{\mu}$ remains invariant whereas its direction evolves together with the deformation of the material filament. Explicit expressions are obtained for the material and spatial derivatives of $\boldsymbol{\mu}$, demonstrating their dependence on the local velocity gradient, transverse deformation, curvature and torsion of the filament. A continuous transition of the vector field from the internal concave kern to the external shell is established, ensuring conservation of the oriented vorticity flux along the same material filament. The relation between the microvortex circulation vector, the vector potential of the flow, and the local helicity is derived. It is shown that the sign of the helicity is determined by the consistency between the direction of longitudinal transport and the sign of the circulation. The proposed formulation provides an integral framework for describing coherent vortex structures and forms a mathematical basis for subsequent investigations of vortex stability, helicity transport, and topological invariants of three-dimensional vortex flows.

1. Introduction.

The mathematical description of vortex structures has traditionally been based on the local fields of velocity and vorticity introduced in the classical works of Helmholtz, Kelvin and Lamb. In this formulation the vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}$$

plays the role of the principal local characteristic of rotational motion, while the circulation

$$\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l}$$

is regarded as an integral invariant associated with a material contour.

Although these quantities provide the foundation of classical vortex dynamics, neither of them simultaneously describes both the magnitude of circulation and its spatial orientation along an evolving vortex filament. The circulation is a scalar associated with a closed contour, whereas the vorticity is a differential field depending on local derivatives of the velocity. During deformation of three-dimensional coherent vortices, it becomes desirable to introduce an integral quantity attached directly to the material filament and evolving together with it.

The present work introduces such a quantity in the form of the **microvortex circulation vector**

$$\boldsymbol{\mu} = \Gamma\,\mathbf{T},$$

where Γ denotes the circulation around the transverse cross-section of a vortex filament and $\mathbf{T}$ is the unit tangent vector of the material curve. This vector represents the oriented flux of vorticity through the local cross-section of the filament and therefore combines the integral circulation with the geometric orientation of the vortex line. Unlike the vorticity field, the vector $\boldsymbol{\mu}$ is attached to the material filament itself and naturally follows its deformation.

The analysis is carried out for a unified ellipsoidal vortex possessing a central concave helical kern [1-2]. The proposed geometry differs from classical cylindrical vortex-core models by introducing a narrowing funnel-shaped kern whose helical filaments continuously emerge into an external quasi-meridional shell. This geometry provides a convenient framework for studying the transport of circulation from the internal rotating region toward the outer vortex envelope while preserving the orientation of individual material filaments.

Within this framework explicit expressions are derived for the material and spatial evolution of the microvortex circulation vector. The governing equations show that in an ideal incompressible fluid the magnitude of $\boldsymbol{\mu}$ remains constant as a consequence of Kelvin's circulation theorem [3-5], whereas its direction changes due to transverse deformation and local rotation of the surrounding medium. The spatial derivatives of the vector are expressed through the curvature and torsion of the vortex filament, establishing a direct connection between geometry and integral rotational dynamics.

The paper also establishes the relationship between the microvortex circulation vector, the vector potential of the velocity field, and the local hydrodynamic helicity. These results demonstrate that μ serves as the principal integral characteristic of the rotational state of an individual vortex filament and provides a natural mathematical language for describing coherent vortex structures whose geometry evolves continuously in space and time.

2. Definition of the Microvortex Circulation Vector $\boldsymbol{\mu}$.

To provide an integral description of the rotational state of the central concave kern, we introduce the microvortex circulation vector

$$\boldsymbol{\mu}_\alpha(s,t) = \Gamma_\alpha(s,t)\mathbf{T}_\alpha(s,t),$$

which combines the magnitude of the local circulation with the orientation of the corresponding vortex filament. Unlike the vorticity $\boldsymbol{\omega}$, which is a local field quantity, the vector $\boldsymbol{\mu}_\alpha$ is an integral characteristic representing the flux of vorticity through a small transverse cross-section of a vortex tube. For the concave helical kern, this definition is applied to the entire family of internal filaments, since their tangent directions vary continuously along the narrowing funnel.

Let an individual filament of the kern be represented by the regular material curve $\boldsymbol{\gamma}_{K,\alpha}(s,t)$, where s is the arc-length parameter, t is time, and α labels the individual filaments. The corresponding unit tangent vector is

$$\mathbf{T}_\alpha(s,t) = \frac{\partial_s \boldsymbol{\gamma}_{K,\alpha}}{\left|\partial_s \boldsymbol{\gamma}_{K,\alpha}\right|}.$$

Around this filament, choose a small closed material contour $C_\alpha(s,t)$, lying in the plane normal to $\mathbf{T}_\alpha$. The local circulation is defined by

$$\Gamma_\alpha(s,t) = \oint_{C_\alpha(s,t)} \mathbf{v}\cdot d\mathbf{l}.$$

By Stokes' theorem,

$$\Gamma_\alpha(s,t) = \int_{S_\alpha(s,t)} \boldsymbol{\omega}\cdot\mathbf{n}_\alpha\, dS, \qquad \boldsymbol{\omega} = \nabla\times\mathbf{v},$$

where S_α is the surface bounded by the contour C_α, and its unit normal is chosen consistently with the tangent direction, $\mathbf{n}_\alpha = \mathbf{T}_\alpha$.

The microvortex circulation vector is defined by

$$\boxed{\boldsymbol{\mu}_\alpha = \Gamma_\alpha \mathbf{T}_\alpha}.$$

Its magnitude is $|\boldsymbol{\mu}_\alpha| = |\Gamma_\alpha|$, while its direction coincides with the local orientation of the vortex filament. The physical dimension of this vector is the same as that of circulation,

$$[\boldsymbol{\mu}] = \frac{L^2}{T}.$$

Therefore $\boldsymbol{\mu}$ should not be identified with the vorticity $\boldsymbol{\omega}$, whose dimension is T^{-1}.

In the thin-vortex-tube approximation, where the vorticity is nearly uniform over the cross-section and is predominantly aligned with the filament,

$$\boldsymbol{\omega}_\alpha \simeq \omega_{\parallel,\alpha}\mathbf{T}_\alpha,$$

one has $\Gamma_\alpha \simeq \omega_{\parallel,\alpha}A_\alpha$, where A_α denotes the cross-sectional area of the filament. Consequently,

$$\boldsymbol{\mu}_\alpha \simeq A_\alpha \boldsymbol{\omega}_\alpha.$$

Thus, the vector $\boldsymbol{\mu}_\alpha$ represents the oriented flux of vorticity through the local transverse cross-section of the vortex tube.

For a filament of the concave kern, parameterized by the longitudinal coordinate σ, the non-unit tangent vector is given by

$$\partial_\sigma \boldsymbol{\gamma}_{K,\alpha} = R_K{}'(\sigma)\mathbf{e}_r + R_K(\sigma)q_K(\sigma)\mathbf{e}_\varphi + z_K{}'(\sigma)\mathbf{e}_z.$$

Introduce the notation

$$\Lambda_K(\sigma) = \sqrt{(R_K{}'(\sigma))^2 + R_K^2(\sigma)q_K^2(\sigma) + (z_K{}'(\sigma))^2}.$$

Then the unit tangent vector is

$$\mathbf{T}_{K,\alpha} = \frac{R_K{}'\mathbf{e}_r + R_K q_K \mathbf{e}_\varphi + z_K{}'\mathbf{e}_z}{\Lambda_K},$$

and the microvortex circulation vector inside the kern is written as

$$\boldsymbol{\mu}_{K,\alpha} = \frac{\Gamma_{K,\alpha}}{\Lambda_K}\left(R_K{}'\mathbf{e}_r + R_K q_K \mathbf{e}_\varphi + z_K{}'\mathbf{e}_z\right).$$

Its components in the natural local basis are

$$\mu_r = \Gamma_{K,\alpha}\frac{R_K{}'}{\Lambda_K}, \qquad \mu_\varphi = \Gamma_{K,\alpha}\frac{R_K q_K}{\Lambda_K}, \qquad \mu_z = \Gamma_{K,\alpha}\frac{z_K{}'}{\Lambda_K}.$$

The radial component reflects the progressive narrowing of the funnel, the azimuthal component characterizes the helical orientation of the filament, and the longitudinal component describes the transport of circulation from the northern inlet toward the central outlet. The sign of the azimuthal component is determined by the product $\Gamma_{K,\alpha}q_K$, which expresses the consistency between the **dynamic circulation** and the **geometric orientation** of the helical filament.

For a vortex of the first kind, the signs of the circulation and the winding density are assumed to be consistent for all internal filaments:

$$\Gamma_{K,\alpha}\Gamma_{K,\beta} > 0, \qquad q_{K,\alpha}q_{K,\beta} > 0$$

for any pair of filaments α and β. Consequently, the azimuthal components of the microvortex circulation vectors have a common orientation:

$$\mu_{\varphi,\alpha}\mu_{\varphi,\beta} > 0.$$

The chirality of the vector $\boldsymbol{\mu}_\alpha$ is determined by the sign of the circulation for a fixed orientation of the material filament:

$$\chi_{\mu,\alpha} = \operatorname{sgn}\Gamma_\alpha.$$

The direction of traversal of the transverse contour is chosen according to the **right-hand rule** with respect to the tangent vector $\mathbf{T}_\alpha$ [1]. Accordingly, $\chi_{\mu,\alpha} = +1$ corresponds to positive circulation with respect to the oriented local frame, whereas $\chi_{\mu,\alpha} = -1$ corresponds to the opposite orientation.

3. Material Evolution of the Vector.

In an ideal incompressible barotropic fluid acted upon only by conservative body forces, the circulation around a material contour is conserved:

$$\frac{D\Gamma_\alpha}{Dt} = 0. (**)$$

The material derivative of the microvortex circulation vector is

$$\frac{D\boldsymbol{\mu}_\alpha}{Dt} = \frac{D\Gamma_\alpha}{Dt}\mathbf{T}_\alpha + \Gamma_\alpha\frac{D\mathbf{T}_\alpha}{Dt}.$$

If the circulation is conserved (**), then

$$\frac{D\boldsymbol{\mu}_\alpha}{Dt} = \Gamma_\alpha\frac{D\mathbf{T}_\alpha}{Dt}.$$

Thus, the magnitude of the vector remains constant, whereas its direction changes together with the tangent direction of the material filament. Since $\mathbf{T}_\alpha\cdot\mathbf{T}_\alpha = 1$, it follows that

$$\mathbf{T}_\alpha\cdot\frac{D\mathbf{T}_\alpha}{Dt} = 0.$$

Hence,

$$\boldsymbol{\mu}_\alpha \cdot \frac{D\boldsymbol{\mu}_\alpha}{Dt} = 0$$

whenever the circulation is conserved, and therefore

$$\frac{D}{Dt}|\boldsymbol{\mu}_\alpha|^2 = 0.$$

Consequently, $|\boldsymbol{\mu}_\alpha(t)| = |\boldsymbol{\mu}_\alpha(0)|$ as long as the circulation conservation law remains valid.

For a material filament, the evolution of the unit tangent vector is expressed through the velocity-gradient tensor:

$$\frac{D\mathbf{T}_\alpha}{Dt} = (\mathbf{I} - \mathbf{T}_\alpha \otimes \mathbf{T}_\alpha)(\nabla\mathbf{v})\mathbf{T}_\alpha.$$

Consequently,

$$\frac{D\boldsymbol{\mu}_\alpha}{Dt} = \Gamma_\alpha(\mathbf{I} - \mathbf{T}_\alpha \otimes \mathbf{T}_\alpha)(\nabla\mathbf{v})\mathbf{T}_\alpha.$$

The operator

$$\mathbf{P}_\perp = \mathbf{I} - \mathbf{T}_\alpha \otimes \mathbf{T}_\alpha$$

is the orthogonal projector onto the plane normal to the filament. Therefore, the change in the direction of $\boldsymbol{\mu}_\alpha$ is determined by the transverse component of the local deformation induced by the velocity field.

Let the velocity-gradient tensor be decomposed into its symmetric and antisymmetric parts: $\nabla\mathbf{v} = \mathbf{D} + \mathbf{W}$, where

$$\mathbf{D} = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathsf{T}})$$

is the rate-of-strain tensor, and

$$\mathbf{W} = \frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathsf{T}})$$

is the local rotation tensor. Then

$$\frac{D\mathbf{T}_\alpha}{Dt} = \mathbf{P}_\perp\mathbf{D}\mathbf{T}_\alpha + \mathbf{P}_\perp\mathbf{W}\mathbf{T}_\alpha.$$

Since $\mathbf{W}$ is antisymmetric

$$\mathbf{T}_\alpha \cdot \mathbf{W}\mathbf{T}_\alpha = 0\,,$$

and therefore

$$\mathbf{P}_\perp\mathbf{W}\mathbf{T}_\alpha = \mathbf{W}\mathbf{T}_\alpha.$$

The tangent-evolution equation may consequently be written as

$$\frac{D\mathbf{T}_\alpha}{Dt} = (\mathbf{I} - \mathbf{T}_\alpha \otimes \mathbf{T}_\alpha)\,\mathbf{D}\mathbf{T}_\alpha + \mathbf{W}\mathbf{T}_\alpha.$$

For conserved circulation, the evolution equation for the microvortex circulation vector becomes

$$\frac{D\boldsymbol{\mu}_\alpha}{Dt} = \mathit{\Gamma}_\alpha\mathbf{P}_\perp\mathbf{D}\mathbf{T}_\alpha + \mathit{\Gamma}_\alpha\mathbf{P}_\perp\mathbf{W}\mathbf{T}_\alpha.$$

Since, for any vector $\mathbf{a}$,

$$\mathbf{W}\,\mathbf{a} = \tfrac{1}{2}\boldsymbol{\omega} \times \mathbf{a},$$

and the vector $\boldsymbol{\omega} \times \mathbf{T}_\alpha$ is already orthogonal to $\mathbf{T}_\alpha$, we obtain

$$\boxed{\frac{D\boldsymbol{\mu}_\alpha}{Dt} = \Gamma_\alpha[(\mathbf{I} - \mathbf{T}_\alpha \otimes \mathbf{T}_\alpha)\mathbf{D}\mathbf{T}_\alpha + \mathbf{W}\mathbf{T}_{\boldsymbol{\alpha}}]}.$$

Using $\boldsymbol{\mu}_\alpha = \Gamma_\alpha\mathbf{T}_\alpha$ this equation can equivalently be expressed directly in terms of $\boldsymbol{\mu}_\alpha$:

$$\boxed{\frac{D\boldsymbol{\mu}_\alpha}{Dt} = \left(\mathbf{I} - \frac{\mathbf{T}_\alpha \otimes \mathbf{T}_\alpha}{|\boldsymbol{\mu}_\alpha|^2}\right)\mathbf{D}\boldsymbol{\mu}_\alpha + \mathbf{W}\boldsymbol{\mu}_\alpha}$$

provided that $D\Gamma_\alpha/Dt = 0$. The first term describes the turning of the filament caused by the transverse deformation of the flow, whereas the second term represents its rotation induced by the local rotational motion of the surrounding medium.

Thus, the antisymmetric part $\mathbf{W}$ rotates the microvortex circulation vector locally, whereas the symmetric part $\mathbf{D}$ changes its direction through transverse stretching and deformation while preserving its magnitude when the circulation remains constant.

In the stationary geometric description, the derivative of the microvortex circulation vector $\boldsymbol{\mu}_\alpha$ with respect to the arc length is

$$\frac{d\boldsymbol{\mu}_\alpha}{ds} = \frac{d\Gamma_\alpha}{ds}\mathbf{T}_\alpha + \Gamma_\alpha\frac{d\mathbf{T}_\alpha}{ds}.$$

According to the Frenet formula, $\frac{d\mathbf{T}_\alpha}{ds} = \kappa_\alpha\mathbf{N}_\alpha$, where κ_α is the curvature and $\mathbf{N}_\alpha$, is the principal normal vector. Therefore,

$$\frac{d\boldsymbol{\mu}_\alpha}{ds} = \frac{d\Gamma_\alpha}{ds}\mathbf{T}_\alpha + \Gamma_\alpha\kappa_\alpha\mathbf{N}_\alpha.$$

If the circulation remains constant along the filament, $\frac{d\mathit{\Gamma}_\alpha}{ds} = 0$, then

$$\boxed{\frac{d\boldsymbol{\mu}_\alpha}{ds} = \Gamma_\alpha\kappa_\alpha\mathbf{N}_\alpha}.$$

Thus, the spatial derivative of the microvortex circulation vector is directed along the principal normal and is proportional to the curvature of the filament. Along a straight segment, where $\kappa_\alpha = 0$, the direction of $\boldsymbol{\mu}_\alpha$ remains unchanged.

The second spatial derivative for the case of conserved circulation is

$$\frac{d^2\boldsymbol{\mu}_\alpha}{ds^2} = \Gamma_\alpha\frac{d}{ds}(\kappa_\alpha\mathbf{N}_\alpha).$$

Using the Frenet formula $\frac{d\mathbf{N}_\alpha}{ds} = -\kappa_\alpha\mathbf{T}_\alpha + \tau_\alpha\mathbf{B}_\alpha$, we obtain

$$\boxed{\frac{d^2\boldsymbol{\mu}_\alpha}{ds^2} = \Gamma_\alpha(-\kappa_\alpha^2\mathbf{T}_\alpha + \kappa_\alpha'\mathbf{N}_\alpha + \kappa_\alpha\tau_\alpha\mathbf{B}_\alpha)}.$$

The tangential component is determined by the square of the curvature, the normal component by the variation of the curvature, and the binormal component by the combined action of curvature and torsion. Consequently, the spatial evolution of the microvortex circulation vector $\boldsymbol{\mu}_\alpha$ directly reflects the geometry of the helical filament.

Within the concave kern, both the curvature and the torsion depend on the functions $R_K(\sigma)$, $z_K(\sigma)$, $q_K(\sigma)$. Since, in the present model, $R_K{}'(\sigma) \neq 0$, $q_K{}'(\sigma) \neq 0$, the filament is not an ordinary cylindrical helix of constant radius and pitch. As the filament approaches the central outlet, the geometric direction of the vector $\boldsymbol{\mu}_\alpha$ changes together with the tangent vector, whereas its magnitude may remain constant provided that the circulation is conserved.

In a viscous fluid, exact conservation of circulation is generally violated. For a Newtonian fluid, the material derivative of the circulation around a material contour can be expressed as

$$\frac{D\Gamma_\alpha}{Dt} = \nu\oint_{C_\alpha}\mathit{\Delta}\,\mathbf{v}\cdot d\mathbf{l} + \oint_{C_\alpha}\mathbf{f}_{\mathrm{nc}}\cdot d\mathbf{l},$$

where ν is the kinematic viscosity and $\mathbf{f}_{\mathrm{nc}}$ denotes the nonconservative body force per unit mass. Thus, we have

$$\frac{D\boldsymbol{\mu}_\alpha}{Dt} = \frac{D\Gamma_\alpha}{Dt}\mathbf{T}_\alpha + \Gamma_\alpha\frac{D\mathbf{T}_\alpha}{Dt}.$$

The first term changes the magnitude of $\boldsymbol{\mu}_\alpha$, whereas the second changes its direction. In the metastable regime, it is assumed that the relative variation of the circulation is small on the characteristic time scale of the geometric evolution:

$$\left|\frac{1}{\Gamma_\alpha}\frac{D\Gamma_\alpha}{Dt}\right| \ll \left|\frac{D\mathbf{T}_\alpha}{Dt}\right|.$$

Under this condition, the vector $\boldsymbol{\mu}_\alpha$ primarily rotates together with the filament, while its magnitude varies slowly and preserves its sign.

4. Spatial Evolution and Continuity of the Vector.

When the filament leaves the central kern, the continuity condition for the microvortex circulation vector is introduced:

$$\boldsymbol{\mu}_{S,\alpha}(0) = \boldsymbol{\mu}_{K,\alpha}(L_K).$$

For nonzero circulation, this condition is satisfied by the simultaneous fulfillment of

$$\Gamma_{S,\alpha}(0) = \Gamma_{K,\alpha}(L_K) \text{ и } \mathbf{T}_{S,\alpha}(0) = \mathbf{T}_{K,\alpha}(L_K).$$

Consequently, the tangential departure of a material filament from the surface of the funnel is accompanied by the continuous transfer of its integral circulation characteristic into the outer region of the vortex. For a smoothly oriented family of filaments, one may introduce the distributed field

$$\boldsymbol{\mu}(\mathbf{r},t) = \Gamma(\mathbf{r},t)\mathbf{T}(\mathbf{r},t).$$

In the thin-vortex-tube approximation, $\boldsymbol{\mu} \simeq A_f\boldsymbol{\omega}$, where A_f denotes the area of the local cross-section of an elementary vortex tube. Under incompressible deformation, the volume of a small material element of the vortex tube is conserved:

$$A_f(s,t)\,dl(s,t) = A_f(s,0)\,dl(s,0).$$

If the vorticity flux is conserved, $\Gamma = A_f\omega_\parallel =$ const, then

$$\omega_\parallel = \frac{\Gamma}{A_f}.$$

Therefore, a reduction of the cross-sectional area is accompanied by an increase in the longitudinal vorticity, although the magnitude of the integral vector $\boldsymbol{\mu}$ remains equal to the conserved circulation.

For a vortex of the first kind, the circulations of all filaments are required to have the same sign:

$$\Gamma_\alpha\Gamma_\beta > 0.$$

The local vectors $\boldsymbol{\mu}_\alpha$ are not required to be mutually parallel, since the filaments occupy different spatial positions and possess different tangent directions. Hence, the global condition

$$\boldsymbol{\mu}_\alpha \cdot \boldsymbol{\mu}_\beta > 0$$

is meaningful only after the compared vectors have been parallel transported to a common point, or for sufficiently nearby filaments for which the effects of spatial curvature can be neglected. Γ_α with respect to the chosen orientation of the material foliation.

The integral circulation vector of the entire kern may be defined as the total vorticity flux through its transverse cross-section:

$$\boldsymbol{\mu}_K(\sigma,t) = \int_{S_K(\sigma,t)} \boldsymbol{\omega}\,(\mathbf{r},t)\,d\mathbf{S},$$

where $d\mathbf{S} = \mathbf{n}_K\,dS$ is oriented along the local longitudinal direction of the kern. If the vorticity is nearly parallel to the filament tangent and varies only weakly across the cross-section, then

$$\boldsymbol{\mu}_K \simeq \Gamma_K\mathbf{T}_K.$$

5. Relation to the Vector Potential and Helicity.

The connection between the microvortex circulation vector and the vector potential of the velocity field is established through the relations [1-2] is established through the relations

$$\mathbf{v} = \nabla\times\mathbf{A}, \qquad \boldsymbol{\omega} = \nabla\times\nabla\times\mathbf{A}.$$

Therefore,

$$\boldsymbol{\mu}_K = \int_{S_K} (\nabla\times\nabla\times\mathbf{A})\,d\mathbf{S},$$

while the circulation is

$$\Gamma_K = \oint_{C_K} (\nabla\times\mathbf{A})\cdot d\mathbf{l}.$$

Thus, the vector potential determines the local velocity and vorticity fields, whereas $\boldsymbol{\mu}_K$ characterizes the integral flux of vorticity through the transverse cross-section of the kern.

The microvortex circulation vector also appears in the local expression for the hydrodynamic helicity. If, within a thin vortex filament, $\mathbf{v} \simeq U_\parallel\mathbf{T}$, and $\boldsymbol{\omega} \simeq \boldsymbol{\mu}/A_f$, then

$$h = \mathbf{v}\cdot\boldsymbol{\omega} \simeq \frac{U_\parallel}{A_f}\mathbf{T}\cdot\boldsymbol{\mu}.$$

Since $\boldsymbol{\mu} = \Gamma\,\mathbf{T}$, it follows that

$$h \simeq \frac{U_\parallel\Gamma}{A_f}.$$

Therefore, the sign of the local helicity is determined by the consistency between the direction of the longitudinal transport and the sign of the circulation.

Thus, the vector

$$\boxed{\boldsymbol{\mu} = \Gamma\,\mathbf{T}}$$

constitutes the principal integral characteristic of the rotational state of an individual vortex filament. Its magnitude is equal to the local circulation, while its direction coincides with the tangent to the material curve. The spatial derivative of $\boldsymbol{\mu}$ is governed by the curvature and torsion of the filament, whereas its material derivative is determined by the transverse deformation of the flow and the local rotation of the surrounding fluid. In the ideal regime, the magnitude of the vector is conserved, whereas in a viscous metastable regime it may vary gradually without a reversal of the circulation sign. The continuity of $\boldsymbol{\mu}$ across the outlet cross-section ensures the transfer of the oriented vorticity flux from the internal concave kern to the outer continuation of the same material filament.

6. Fundamental Theorem on the Microvortex Circulation Vector.

The principal properties of the microvortex circulation vector may be summarized in the following theorem.

Theorem 1 (Existence and Evolution of the Microvortex Circulation Vector).

Let $\gamma(s,t)$ be a smooth material vortex filament transported by an incompressible barotropic flow. Assume that the circulation $\Gamma(s,t)$ is defined around every sufficiently small material contour normal to the filament. Then the vector $\boldsymbol{\mu}(s,t) = \Gamma(s,t)\,\mathbf{T}(s,t)$, where $\mathbf{T} = \frac{\partial_s\boldsymbol{\gamma}}{|\partial_s\boldsymbol{\gamma}|}$ is the unit tangent vector, possesses the following properties.

1. The magnitude of the vector equals the local circulation, $|\boldsymbol{\mu}| = |\Gamma|$.

2. The direction of the vector coincides with the tangent direction of the material filament.

3. If Kelvin's circulation theorem holds, $\frac{D\Gamma}{Dt} = 0$, then $\frac{D\boldsymbol{\mu}}{Dt} = \Gamma\,(\mathbf{I} - \mathbf{T} \otimes \mathbf{T})(\nabla\mathbf{v})^T$, and therefore $\frac{D|\boldsymbol{\mu}|^2}{Dt} = 0$.

4. The spatial derivative of the vector is determined by the local geometry of the filament: $\frac{d\boldsymbol{\mu}}{ds} = \Gamma\,\kappa\,\mathbf{N}$, provided that $\frac{d\Gamma}{ds} = 0$.

5. The second spatial derivative satisfies $\frac{d^2\boldsymbol{\mu}}{ds^2} = \Gamma(-\kappa^2\mathbf{T} + \kappa'\mathbf{N} + \kappa\tau\mathbf{B})$, thus depending explicitly on the curvature and torsion of the vortex filament.

6. If the filament passes continuously from the concave kern into the outer shell, $\boldsymbol{\mu}_S(0) = \boldsymbol{\mu}_K(L_K)$, then the oriented vorticity flux is continuously transported along the same material filament.

Proof. The first two statements follow immediately from the definition $\boldsymbol{\mu} = \Gamma\mathbf{T}$.

The evolution equation is obtained by differentiating this expression with respect to time and using Kelvin's theorem, $\frac{D\Gamma}{Dt} = 0$, together with the transport equation for the tangent vector,

$$\frac{D\mathbf{T}}{Dt} = (\mathbf{I} - \mathbf{T} \otimes \mathbf{T})(\nabla\mathbf{v})^T.$$

Since $\mathbf{T} \cdot \frac{D\mathbf{T}}{Dt} = 0$, the magnitude of the vector remains unchanged,

$$\frac{D|\boldsymbol{\mu}|^2}{Dt} = 2\boldsymbol{\mu} \cdot \frac{D\boldsymbol{\mu}}{Dt} = 0.$$

The spatial derivatives follow directly from the Frenet formulas

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \qquad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}.$$

Finally, the continuity condition at the outlet cross-section guarantees that both the circulation and the tangent orientation remain continuous across the interface, implying continuity of the microvortex circulation vector itself. Therefore, the oriented vorticity flux is preserved along the same material vortex filament. ■

7. Conclusion.

The present work introduces the microvortex circulation vector

$$\boldsymbol{\mu} = \Gamma\,\mathbf{T}$$

as a new integral characteristic of coherent vortex filaments. Unlike the classical vorticity field, the proposed vector combines the magnitude of the local circulation with the geometric orientation of the material filament and therefore provides an oriented description of the rotational state of the flow.

Explicit evolution equations have been obtained for both the material and spatial derivatives of the vector. They demonstrate that the geometry of the filament, through its curvature and torsion, together with the local velocity-gradient tensor, completely determines the evolution of the direction of the microvortex circulation vector, while its magnitude remains conserved under Kelvin's circulation theorem and varies only gradually in a viscous metastable regime.

The established continuity of the vector field across the outlet section of the concave kern provides a mathematical description of the transfer of the oriented vorticity flux from the internal rotating region to the external continuation of the same material filament. The obtained relation between the microvortex circulation vector, the vector potential and the local hydrodynamic helicity demonstrates that the sign of the helicity is governed by the consistency between the longitudinal transport direction and the circulation sign.

Thus, the microvortex circulation vector forms a natural integral variable for the mathematical description of coherent vortex structures. The proposed formulation provides a foundation for future investigations of vortex stability, helicity transport, spectral properties of perturbations, and topological invariants associated with the linking and twisting of three-dimensional vortex filaments.

References:

1. Milyute E., Milyuvene V., Milyus A. Geometric and Hydrodynamics Model of a unified ellipsoidal Vortex with a Central Concave Kern and Intrinsic Helicity, NJD-iScience, 2026, **186**

2. Milyute E. Derivative of the circulation invariant and preservation of the diffeomorphism of vortex filaments. Book of Abstracts, 2026 International Congress of Mathematicians, Philadelphif, Pennsylvania, USA

3. Kelvin W. T. On Vortex Motion. *Transactions of the Royal Society of Edinburgh*. 1867. **Vol. 25**. 217–260.

4. Lamb H. *Hydrodynamics*. 6th ed. Cambridge University Press, 1932.

5. Moffatt H. K. *Helicity and singular structures in fluid dynamics*. PNAS,. 2014, **111**, Nr. 10, 3663-3670.